Astronomy 3602: Homework #3

Due Wednesday, October 27

Problem 1: Loitering Universe (25 points).

Suppose we lived in a positively curved loitering universe (as discussed in Ryden § 5.4.3) that is currently expanding, with $\Omega_{m,0} = 0.25$. What is the value of $\Omega_{\Lambda,0}$? What is the redshift at which you would expect to see many galaxies (i.e. the redshift at which the Universe spent an unusually long time)?

Problem 2: Metric with Positive Curvature from Embedding (30 points).

As mentioned in class, it is possible to derive the spatial part of the FRW metric with positive curvature by embedding a 3-sphere in flat, four-dimensional Euclidean space. Show this explicitly, by considering the distance in 4D Cartesian coordinates, $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$, and computing the 3D metric that results from restricting the distance to lie on the surface of a 3-sphere. Hint: the equation of the 3-sphere is $x^2 + y^2 + z^2 + w^2 = R^2$, where R=constant; you can use this equation to eliminate w from ds^2. Show that after a coordinate transformation from Cartesian to spherical coordinates, you recover the spatial part of the FRW metric $ds^2 = (1 - kr^2)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$. How is k related to R ?

Problem 3: Angular Sizes in Curved Space (15 points).

Suppose we were two-dimensional beings, living in a positively curved two-dimensional universe, represented by the surface of a sphere with radius R. An object whose size is $dl \ll R$ is placed at a distance r from you (remember, all distances are measured on the surface of the sphere). What is the angular size $d\theta$ extended by this object, from your point of view? Explain the behavior of $d\theta$ as $r \to \pi R$.

Problem 4: Seeing Around the Universe (30 points).

A photon is emitted at the time of the Big Bang in a universe that contains only non-relativistic matter, and has $k > 0$. Here we will show that the photon travels precisely all the way around the universe by the time of the "Big Crunch".

(a) Recall that the spatial part of the FRW metric

$$ds^2 = -c^2dt^2 + a(t)^2 \left[dr^2 + R_0^2 \sin^2(r/R_0)(d\theta^2 + \sin^2\theta d\phi)\right] \tag{1}$$

for a positively curved space-time describes a sphere of radius R_0, where the radius of curvature R_0 is related to k by $k = c^2/R_0^2$. What is the proper circumference of the universe at cosmic time t? **[10 points]**

(b) Choose the coordinate system such that the photon is emitted at $t = 0$ at the origin $r = 0$, and travels along the geodesic with $\theta = \phi = 0$. Write down the expression for its proper distance from the origin at cosmic time t as an integral over t. Divide this expression by your answer in (a) to obtain the fraction $f(t)$ of the circumference covered by the photon by time t. **[10 points]**

(c) Use the parametric solution given in Problem Set #2,

$$\begin{aligned} a(\theta) &= \frac{4\pi G\rho_0}{3k}(1 - \cos\theta) \\ t(\theta) &= \frac{4\pi G\rho_0}{3k^{3/2}}(\theta - \sin\theta) \end{aligned}$$

to convert the integral over t in (b) to an integral over θ, and find the function $f(\theta)$. Show that $f(\theta = \pi) = 1/2$ and $f(\theta = 2\pi) = 1$. **[10 points]**