Astronomy 3602: Homework #2

Due Wednesday, October 13

Problem 1: Friedmann Equation for $k > 0$ (25 points).

The full Friedmann equation in general is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\epsilon - \frac{k}{a^2}. \quad (1)$$

Consider the case when the Universe contains only non-relativistic matter, so that the energy density $\epsilon = \rho = \rho_0/a^3$. Note that we have set the speed of light $c = 1$, and ρ_0 is the present–day mass density of the universe (more generally, the subscript $_0$ on any parameter refers to the value of the parameter at the present epoch; hence t_0 is the present age of the Universe, and a_0 is the present–day scale factor). According to this convention, at the present day, $t = t_0$, we have $a_0 \equiv a(t_0) = 1$.

(a) Demonstrate that the following parametric solution

$$a(\theta) = \frac{4\pi G\rho_0}{3k}(1 - \cos\theta)$$
$$t(\theta) = \frac{4\pi G\rho_0}{3k^{3/2}}(\theta - \sin\theta)$$

solves this equation. Here θ is a parameter that runs from $0 \leq \theta \leq 2\pi$. (Hint: use the chain rule.) [**15 points**]

(b) Make a plot of both a and t as functions of the parameter θ. With the help of these, plot the evolution of the scale factor $a(t)$. [**10 points**]

Problem 2: The Age of a Single-Component Universe (35 points).

(a) Show that if one assumes $k = 0$, then using the Friedmann equation, together with a measurement of the Hubble constant, one can derive t_0, the age of the Universe. Recall that the Hubble constant is defined as $H_0 = (\dot{a}/a)\mid_{t=t_0}$. What is the value of t_0 if $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$? Express your answer in Gyr. [**5 points**]

(b) Consider a Universe with $k > 0$. For the fixed value of $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, is the current age of this Universe longer or shorter than it would be for $k = 0$? Make an argument by comparing your sketch for the solution for $a(t)$ from Problem 1 to the solution

$a \propto t^{2/3}$ for the case of $k = 0$. (Hint: you have to match the two solutions $a(t)$ for $k = 0$ and for $k > 0$ to have the same amplitude, a_0, and derivative, H_0, at $t = t_0$). [**6 points**]

(c) Next, be more quantitative, and use the solution for the $k > 0$ universe from Problem 1 to derive an expression for t_0 that depends only on H_0 and θ_0 (but not on k). Given that the Universe is expanding now, what are the possible values of θ_0? What is the corresponding allowed range for t_0. (Hint: you will have to use L'Hopital's rule repeatedly). [**12 points**]

(d) Use the solution for the $k > 0$ universe from Problem 1, and derive an expression for $t_{\rm max}$, the time at which the Universe reaches maximum expansion, in terms of t_0 and θ_0. Suppose we know the age of the universe is at least $(\pi/2 - 1)H_0^{-1} = 8.15$ Gyr (e.g. from the ages of the oldest stars). What is the shortest time $(t_{\rm max} - t_0)$ we have to wait until the Universe stops expanding? [**12 points**]

Problem 3: Mixed Models with Matter + Radiation (20 points).

The present–day temperature of the cosmic microwave background is measured to be $T_{\rm r,0} = 2.73$ K. The present–day energy density in this radiation field is therefore $\epsilon_{\rm r,0} = \alpha T_{\rm r,0}^4$ (with $\alpha = 7.56 \times 10^{-16}\ {\rm J\,m^{-3}\,K^{-4}}$ for black–body radiation). The present–day mass–density of the universe is measured to be approximately $\rho_{\rm m,0} = 3 \times 10^{-27}\ {\rm kg\,m^{-3}}$. As mentioned in class, for non-relativistic matter, the energy density scales as $\epsilon_{\rm m} = \epsilon_{m,0}/a^3$, whereas for radiation, $\epsilon_{\rm r} = \epsilon_{\rm r,0}/a^4$. Throughout this problem, you can assume $k = 0$.

(a) Does matter or radiation contribute the dominant energy density in the present–day universe ? What is ratio of the current energy densities, $\epsilon_{\rm r,0}/\epsilon_{\rm m,0}$? What is the redshift at which the radiation and matter energy density are equal ? [**8 points**]

(b) As shown in class, the scale–factor in a one–component universe with matter only would evolve as $a(t) \propto t^{2/3}$, and in a radiation–only universe, it would evolve as $a(t) \propto t^{1/2}$. Sketch the evolution of $a(t)$ for the universe with both matter and radiation, with the measured ratio of $\epsilon_{\rm r,0}/\epsilon_{\rm m,0}$. Plot $\log a$ vs $\log t$. For a given measured H_0, is this universe older or younger than it would be if one assumed radiation did not exist? [**12 points**]

Problem 4: The Age of Our Universe (20 points).

Consider a flat universe with $k = 0$ that has non-relativistic matter and a cosmological constant, with $\Omega_{\Lambda,0} + \Omega_{m,0} = 1$ (i.e., we will ignore radiation which affects the evolution only at very early times, and has only a small effect on the present age).

(a) Show that the Friedmann equation can be written as

$$\dot{a}^2 = H_0^2 \left[\frac{\Omega_{m,0}}{a} + (1 - \Omega_{m,0}) a^2 \right] \tag{2}$$

[5 points]

(b) Solve the above equation to obtain an explicit relation between the cosmic time t and redshift $(1 + z) \equiv 1/a$. (Hint: you will need the indefinite integral $\int dx (a + bx^2)^{-1/2} = b^{-1/2} \ln[x\sqrt{b} + \sqrt{a + bx^2}]$). **[10 points]**

(c) What is the current age if $\Omega_{m,0} = 0.3$, and H_0 =70 km/s/Mpc? The oldest and most distant known galaxy is called "GN-z11" with a confirmed redshift of $z = 11.1$. What was the age of the universe when the light from this galaxy was emitted ? **[5 points]**