Astronomy GR6001: Problem Set #3

Due in my office by Friday, October 22, 2006

Problem 1 (25 points):

In this problem you will compute the attenuation of radiation traveling through a purely scattering medium, a good approximation for clouds. Approximate such a cloud as a uniform slab, composed of a collection of identical, highly reflective, spherical water droplets with radius R and number density η (# of droplets per unit volume). Measure the vertical distance down through the slab in the z direction, with $z = 0$ the top of the cloud, and $z_{\rm max}$ the base of the cloud. As photons enter the cloud from above, they pin-ball on the water droplets, but never get absorbed. Therefore, eventually each photon exits the cloud: either at the base, directed down, or back at the top, directed up. We want to know what fraction of photons make it all the way through the cloud, to the base. For simplicity, we will consider this problem in one dimension, i.e., scattering occurs only back and forth along the z axis.

(a) What is the optical depth of the cloud τ and the mean free path of the photons λ ?

(b) The incident photons, just above the cloud, have a number flux of photons F_i (per unit time and area). What is the number density n_i of photons (per unit volume) above the cloud, just before entering the cloud?

(c) The random walk of the photons down the cloud can be described by a diffusion equation of the form

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2}, \tag{1}$$

where D is a diffusion coefficient, and $n = n(z)$ is the number density of photons. What are the units of D ? Use dimensional analysis to write an expression for D in terms of the speed of light c, η, and R (hint: the only length-scale in the problem is λ).

(d) Write down the solution to the diffusion equation in steady state, i.e. when $\partial n/\partial t = 0$ everywhere in the cloud. (Hint: you will have two constants of integration.)

(e) To solve for the two constants of integration, we need two boundary conditions, which we can specify by defining the flux of transmitted and reflected photons, respectively, as F_t and F_r. (Hint: note that the total flux of photons at the top of the cloud is then given by $F(z = 0) = F_i - F_r$.). Use the above, plus the fact that the incident flux must be the sum of the transmitted + reflected fluxes, $F_i = F_t + F_r$, to show that the attenuation of the flux at the base of the cloud is

$$\frac{F_t}{F_i} = \frac{2}{2 + \tau}. \tag{2}$$

This is much less steep than the exponential attenuation $\exp(-\tau)$ that would be caused by absorbing clouds of the same optical depth (the sky would then be much darker!).

Problem 2 (25 points):

Find the analytic form of the power spectra, $dW/dAd\omega$, shown in Figures 2.1b, 2.2b, and 2.3b of Rybicki & Lightman, assuming that the electric fields, are given, respectively, by

a Gaussian:

$$E(t) = \exp\left(-\frac{t^2}{2T^2}\right), \tag{3}$$

a truncated sinusoid:

$$E(t) = \sin(\omega_0 t) \tag{4}$$

for $\mid t \mid< T$, and $E(t) = 0$ otherwise, and an exponentially decaying sinusioid,

$$E(t) = \exp(-t/T)\sin(\omega_0 t) \tag{5}$$

for $t > 0$ and $E(t) = 0$ for $t < 0$.

Problem 3 (25 points):

Two radio telescopes, a distance D apart, receive a sinusoidal signal from a distant source, located in a direction at an angle θ to the line connecting the telescopes. Find the cross–correlation function of the two measured electric fields, defined as

$$C_{12}(t) \equiv \langle E_1(t+t')E_2(t')\rangle = \frac{1}{T}\int_0^T dt' E_1(t+t')E_2(t'), \tag{6}$$

where T is taken to be very large. Show that C_{12} has many maxima as a function of time delay t, but that the maximum at $t = (D/c)\cos\theta$ is the same for all frequencies. Thus, for a signal consisting of a range of frequencies, one expects a single maximum at this time delay, which can be used to determine θ accurately.

Problem 4 (25 points):

The Stokes parameters I, Q, U, and V are measured with respect to a coordinate system (x, y) in the plane normal to the waves. In another coordinate system, (x', y'), rotated at an angle ϕ with respect to the original system, the Stokes parameters will be I', Q', U', and V'. Show that

$$\begin{aligned} I' &= I \\ V' &= V \\ Q' &= U \sin 2\phi + Q \cos 2\phi \\ U' &= U \cos 2\phi - Q \sin 2\phi \end{aligned}$$