Problem 1: Perturbation growth in a Contracting Universe (25 points).

Problem 12.1 in Ryden

Problem 2: Horizon Size at Radiation-Matter Equality (25 points).

Problem 12.6 in Ryden.

Problem 3: Isothermal Gas Sphere (25 points).

Consider an overdense initial perturbation that separates from the expansion of the universe, turns around, and collapses to form a spherically symmetric, virialized object.

(a) The collapsed gas is shock-heated to the virial temperature $T_{\text{vir}}$. Assume that $T_{\text{vir}}$ is a constant and independent of radius (this is generally a good assumption because conduction would quickly smooth out spatial temperature gradients). Show that the density profile must then have the scaling $\rho = B/r^2$ for the system to be in equilibrium; i.e. for the pressure gradient force to balance the gravitational force. Find an expression for the coefficient $B$ as a function of $T_{\text{vir}}$. [10 points]

(b) Assume that the object collapsed at redshift $z$, it consists of pure (ionized) hydrogen gas (ignore helium and dark matter), and has a total mass $M_{\text{vir}}$, radius $r_{\text{vir}}$, and a mean density $\bar{\rho} = 18\pi^2\rho_c$. Here $\rho_c = 3H^2/(8\pi G)$ is the critical density at the redshift of collapse, and $H(z)$ is the Hubble parameter at $z$. Derive the virial temperature for a typical galaxy with $M_{\text{vir}} = 10^{12}M_\odot$, $z = 2$, and $H_0 = 72$ km/s/Mpc. How does the virial temperature scale with $M_{\text{vir}}$, $z$, and $H_0$? For simplicity, assume a flat universe with $\Omega_m = 1$. [15 points]

Problem 4: Angular Momentum (25 points).

Consider a uniform sphere of mass $M$ and radius $R$, spinning with a total angular momentum $J$. The spin parameter $\lambda_i = (JE^{1/2})/(GM^{5/2})$ measures the ratio of the angular velocity corresponding to $J$ and the angular velocity required to provide full centrifugal support. Here $E = (3/5)GM^2/R$ is the total energy.

(a) Suppose the sphere is compressed by a factor of $f_c$ into a new uniform sphere with radius $R_c = f_c R$. Assuming that angular momentum is conserved, find the value of the spin parameter $\lambda_c$ after the compression, as a function of $f_c$ and $\lambda_i$. [10 points]
(b) Suppose that only a fraction \( f_b = \Omega_b / (\Omega_m + \Omega_b) \) of the original sphere (by mass) is compressed by a factor of \( f_c \) into a new uniform sphere with radius \( R_c = f_c R \), with the remaining fraction \( 1 - f_b = \Omega_m / (\Omega_m + \Omega_b) \) staying in its original configuration as a sphere of radius \( R \). (This is a simplified picture of baryons cooling and condensing inside a stationary dark matter halo). Assume that the compressed material contains a fraction \( f_b \) of the total initial angular momentum. Find the value of the spin parameter \( \lambda_c \) of the compressed material, as a function of \( f_b, f_c, \) and \( \lambda_i \). [10 points]

(c) Detailed models of non-linear structure formation show that tidal torques cause dark matter (DM) halos to have spin parameters of \( \lambda_i \approx 0.1 \) (far from centrifugal support). The baryons contract radially by a factor \( f_c \) relative to the DM halo until they reach centrifugal support, i.e. \( \lambda_c \approx 1 \). What is the factor \( f_c \) implied by your answer to (b)? This is roughly the expected ratio of the size of a galaxy to its DM halo. (Assume \( \Omega_b = 0.04 \) and \( \Omega_m = 0.3 \) for this problem.). [5 points]