Problem 1: Metric with Positive Curvature from Embedding (20 points).

As mentioned in class, it is possible to derive the spatial part of the FRW metric with positive curvature by embedding a 3-sphere in flat, four-dimensional Euclidean space. Show this explicitly, by considering the distance in 4D Cartesian coordinates, \( ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \), and computing the 3D metric that results from restricting the distance to lie on the surface of a 3-sphere. Hint: the equation of the 3-sphere is \( x^2 + y^2 + z^2 + w^2 = R^2 \), where \( R=\text{constant} \); you can use this equation to eliminate \( w \) from \( ds^2 \). Show that after a coordinate transformation from Cartesian to spherical coordinates, you recover the spatial part of the FRW metric

\[
 ds^2 = (1 - kr^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). 
\]

How is \( k \) related to \( R \)?

Problem 2: Friedmann Equation for \( k > 0 \) (25 points).

The full Friedmann equation in general is

\[
 \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \epsilon - \frac{k}{a^2}, 
\]

(1)

Consider the case when the Universe contains only non-relativistic matter, so that the energy density \( \epsilon = \rho = \rho_0/a^3 \). Note that we have set the speed of light \( c = 1 \), and \( \rho_0 \) is the present–day mass density of the universe (more generally, the subscript \( 0 \) on any parameter refers to the value of the parameter at the present epoch; hence \( t_0 \) is the present age of the Universe, and \( a_0 \) is the present–day scale factor). According to this convention, at the present day, \( t = t_0 \), we have \( a_0 \equiv a(t_0) = 1 \).

(a) Demonstrate that the following parametric solution

\[
 a(\theta) = \frac{4\pi G \rho_0}{3k} (1 - \cos \theta) \\
 t(\theta) = \frac{4\pi G \rho_0}{3k^{3/2}} (\theta - \sin \theta) 
\]

solves this equation. Here \( \theta \) is a parameter that runs from \( 0 \leq \theta \leq 2\pi \). [15 points]

(b) Make a plot of both \( a \) and \( t \) as functions of the parameter \( \theta \). With the help of these, plot the evolution of the scale factor \( a(t) \). [10 points]
Problem 3: The Age of the Universe (35 points).

(a) Show that if one assumes $k = 0$, then using the Friedmann equation, together with a measurement of the Hubble constant, one can derive $t_0$, the age of the Universe. Recall that the Hubble constant is defined as $H_0 = (\dot{a}/a) |_{t=t_0}$. What is the value of $t_0$ if $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$? Express your answer in Gyr. [5 points]

(b) Consider a Universe with $k > 0$. For the fixed value of $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, is the current age of this Universe longer or shorter than it would be for $k = 0$? Make an argument by comparing your sketch for the solution for $a(t)$ from Problem 2 to the solution $a(t) \propto t^{2/3}$ for the case of $k = 0$ and for $k > 0$ to have the same amplitude, $a_0$, and derivative, $H_0$, at $t = t_0$. [6 points]

(c) Next, be more quantitative, and use the solution for the $k > 0$ universe from Problem 2 to derive an expression for $t_0$ that depends only on $H_0$ and $\theta_0$ (but not on $k$). Given that the Universe is expanding now, what are the possible values of $\theta_0$? What is the corresponding allowed range for $t_0$ (hint: you will have to use L’Hopital’s rule repeatedly). [12 points]

(d) Use the solution for the $k > 0$ universe from Problem 2, and derive an expression for $t_{\text{max}}$, the time at which the Universe reaches maximum expansion, in terms of $t_0$ and $\theta_0$. Suppose we know the age of the universe is at least $(\pi/2 - 1)H_0^{-1} = 8.15$ Gyr (e.g. from the ages of the oldest stars). What is the shortest time $(t_{\text{max}} - t_0)$ we have to wait until the Universe stops expanding? [12 points]

Problem 4: Mixed Models with Matter + Radiation (20 points).

The present–day temperature of the cosmic microwave background is measured to be $T_{r,0} = 2.73$ K. The present–day energy density in this radiation field is therefore $\epsilon_{r,0} = \alpha T_{r,0}^4$ (with $\alpha = 7.56 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$ for black–body radiation). The present–day mass–density of the universe is measured to be approximately $\rho_{m,0} = 3 \times 10^{-27} \text{ kg m}^{-3}$. As mentioned in class, for non-relativistic matter, the energy density scales as $\epsilon_m = \epsilon_{m,0}/a^3$, whereas for radiation, $\epsilon_r = \epsilon_{r,0}/a^4$. Throughout this problem, you can assume $k = 0$.

(a) Does matter or radiation contribute the dominant energy density in the present–day universe? What is ratio of the current energy densities, $\epsilon_{r,0}/\epsilon_{m,0}$? What is the redshift at which the radiation and matter energy density are equal? [8 points]

(b) As shown in class, the scale–factor in a one–component universe with matter only would evolve as $a(t) \propto t^{2/3}$, and in a radiation–only universe, it would evolve as $a(t) \propto t^{1/2}$. Sketch the evolution of $a(t)$ for the universe with both matter and radiation, with the measured ratio of $\epsilon_{r,0}/\epsilon_{m,0}$. Plot $\log a$ vs $\log t$. For a given measured $H_0$, is this universe older or younger than it would be if one assumed radiation did not exist? [12 points]