Contents

I Introduction 4

1 Cosmic rays and Supernovae 5

2 Radiation mediated shocks 6

3 SN Shock Breakout 8

4 Organization of the thesis 8

II Cosmic Rays from TRSNe 9

5 Constraints on Galactic sources of CRs 9

5.1 Rates and energetics . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

5.2 Maximum energy . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12

5.3 Energy distribution within the ejecta . . . . . . . . . . . . . . . . . . . . . . . . 13

6 Application to SNe and TRSNe 14

6.1 SNe . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14

6.2 TRSNe . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15

III Radiation Mediated Shocks 19

7 The physics of RMS 19

7.1 Far DS conditions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

7.2 From DS radiation domination to radiation mediation . . . . . . . . . . . . . . 20

7.3 The shock transition width . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

7.4 Immediate DS and subsonic region in RRMS . . . . . . . . . . . . . . . . . . . 23

7.5 Structure of RRMS . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
7.6 Low density shocks .................................................. 24
7.7 The conditions under which RMS may exist .................... 25
7.8 Physical assumptions ................................................. 26

8 Formulation of the numerical problem ......................... 27
  8.1 Equations .................................................................. 27
  8.2 Boundary conditions .................................................. 34

9 The numerical method .................................................. 35
  9.1 Iteration scheme ....................................................... 36
  9.2 Discretization ............................................................ 37
  9.3 Test problems ........................................................... 37

10 Numerical results ....................................................... 38
  10.1 Structure ............................................................... 41
  10.2 Spectrum ............................................................... 46
  10.3 Optical depth due to radiation processes ....................... 47
  10.4 Numerical convergence of the calculations .................... 49

11 Simplified analytic modeling of RRMS structure ............... 53
  11.1 The physical picture .................................................. 55
  11.2 General structure ..................................................... 56
  11.3 High energy photon component beamed in the DS direction .. 59
  11.4 Shocks with $\Gamma_u \to \infty$ ............................................ 60

12 NR RMS revisited ...................................................... 64
  12.1 Numerical results .................................................... 64
  12.2 Comparison with previous work .................................. 65

13 Application to Supernova Shock Breakouts ..................... 66
13.1 A general relation between the breakout energy, velocity, and radius 66
13.2 Breakout Dynamics ................................................. 67
13.3 Breakout X-ray characteristics ..................................... 67
13.4 Implications to recently observed SN X-ray outbursts .......... 68

IV Discussion 71

14 CRs from TRSNe 71

15 RRMS 72

16 Application to Supernova Shock Breakout 75

A Notations used in part § III 76
  A.1 Subscripts, superscripts and miscellanea ............................ 76
  A.2 Symbols ................................................................. 77

B Compton scattering approximation 78
  B.1 Low T ................................................................. 79
  B.2 High T ................................................................. 80

C Speed of sound in matter 80

D Transformations and definitions 81
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Part I

Introduction

Recent years have shown great observational advances in the field of high energy transient events, both Gamma Ray Bursts (GRBs) and SuperNovae (SNe). The discovery of X-ray prompt emission from several core collapse SN explosions and the followup measurements lead to an understanding that the shock emerging through the outer layers of the progenitor, or through the wind surrounding it, may be characterized by mildly relativistic velocities. The discovery of a connection of some X-Ray Flashes (XRFs) and low luminosity GRBs, to core collapse SNe, marks the possibility that some of the XRFs and low luminosity GRBs are in fact produced by the shock breakout of a SN.

High energy Cosmic Rays (CRs) have been measured for many decades now, and rich datum is accumulated teaching us of their energy spectrum, the chemical composition and isotopic composition. The origin of most CRs is widely believed to be acceleration in shocks, of which SN and SN Remnant shocks are usually accepted as the source of CRs.

In this thesis we describe several phenomenological and theoretical advances related to these fast shocks and transient events. We address the possibility of acceleration of high energy CRs in the mildly relativistic shocks emerging from a newly discovered type of SN and derive constraints on the accelerators of CRs (in § II). We study (in § III) the structure of Radiation Mediated Shocks (RMSs), expected to emerge from the surface of an exploding star or contained in a decelerated relativistic jet traversing a heavy layer of stellar material, as some models of GRBs suggest.

1. Cosmic rays and Supernovae

The spectrum of CRs extends from $\sim 10^9$ eV to $\sim 10^{20}$ eV (see Blandford & Eichler 1987; Axford 1994; Nagano & Watson 2000, for reviews). While there is a wide consensus that the sources of CRs of energy $\lesssim 10^{15}$ eV are Galactic SNe, and that the sources of CRs above $10^{19}$ eV are extra-Galactic (possibly GRBs or Active Galactic Nuclei (AGNs), e.g. Waxman 2004c, for a recent review), there is no consensus regarding the sources of cosmic rays in the energy range of $\sim 10^{15}$ eV to $\sim 10^{18}$ eV.

The CR spectrum steepens at $\sim 10^{15}$ eV, the ”knee,” and then extends smoothly to $\sim 10^{18}$ eV. These characteristics disfavor a transition from Galactic to extra-Galactic sources in this energy range, since for such a transition one would naturally expect either a flattening of the spectrum or a marked drop of the flux at the transition energy. In fact, these characteristics suggest that
the same type of sources, or the same class of sources, produces the CRs over the entire energy range of \( \sim 10^9 \) eV to \( \sim 10^{18} \) eV. Indeed, it has been suggested that Galactic SNe are the sources of all CRs in this energy range (e.g. Bell & Lucek 2001). However, common SNRs lack the ability to accelerate particles to energies well above the knee, and the acceleration to energies up to \( 10^{18} \) eV is assumed to be located at shocks produced by extremely energetic SN events. As we will show here, this assumption is problematic, since the resulting spectrum of CRs at energies between the knee and the ankle would be inconsistent with current measurements. We argue that a recently discovered new type of SN events may be the source of these CRs.

Over the past several years a new type of supernova explosions has been discovered, to which we will refer as "trans-relativistic supernovae" (TRSNe). We define TRSNe as supernovae which, unlike ordinary SNe, deposit a significant fraction, \( f_R > 10^{-2} \), of their kinetic energy in mildly relativistic, \( \gamma \beta > 1 \), ejecta (the corresponding fraction for ordinary SNe is typically \( \leq 10^{-7} \); see § 6.2 for a detailed discussion). Such TRSNe have been discovered recently in association with gamma-ray bursts (GRBs). Hereafter we will use the term SNe for ordinary supernovae, which deposit only a small fraction of their kinetic energy in relativistic ejecta, and the term TRSNe for supernova explosions with \( f_R > 10^{-2} \).

Various terms have been used in the literature to describe supernovae associated with GRBs: GRB-SNe, "hypernovae", broad-line type Ic SNe, and others. We have chosen here to use the term "trans-relativistic supernovae" for several reasons. First, it captures the main physical property that characterizes supernovae that may produce the observed \( \sim 10^{15} \) eV to \( \sim 10^{18} \) eV cosmic-ray flux, namely \( f_R > 10^{-2} \). Second, supernovae associated with GRBs may not all be similar to each other. SN2006aj, for example, is qualitatively different from the other GRB associated supernovae. Although there is evidence for this explosion to be characterized by \( f_R > 10^{-2} \), its total kinetic energy is similar to that of ordinary SNe, \( \sim 10^{51} \) erg, while the other GRB associated supernovae appear to be ten times more energetic. Thus, while SN2006aj does not fit into the "hypernovae" (i.e. highly energetic) category, it does fit into the TRSNe category and should be considered as a potential source of \( \sim 10^{15} \) eV to \( \sim 10^{18} \) eV cosmic-rays. Finally, it should be kept in mind that not all TRSNe are necessarily associated with GRBs.

### 2. Radiation mediated shocks

Shocks with high velocities, such that the DownStream (DS) energy density is dominated by radiation rather than thermal energy of matter, are expected to occur in various, optically thick, astrophysical flows. The radiation originating from the DS of such Radiation Mediated Shocks (RMS), manages to diffuse to the upstream and to decelerate the incoming plasma (Weaver 1976), resulting in a non-trivial shock structure with momentum and energy carried in the DS
by the radiation.

Non relativistic RMS are expected to traverse the exploding star mantle/envelope during Core Collapse SN explosions. Observations of RMS breakouts from SNe have long been expected to be observed as thermal X-ray bursts preceding SNe optical emission, and may provide valuable information on the progenitor star properties, including its radius, that may help constrain stellar structure and SNe explosion models. Recent observed X-ray flashes preceding SNe emission, most notably the observed X-ray outburst associated with SN2008D, may be the first RMS break-outs ever to have been observed.

Non relativistic RMS were studied in detail in (Weaver 1976), using the diffusion approximation to treat the transport of the radiation, which was described by an effective temperature and photon density. This made the solution of the equations feasible, and was argued to be applicable for low enough velocities ($\beta < 0.45$). In addition, various relativistic effects were neglected.

Recently, two classes of sources have been identified, where Relativistic RMS (RRMS) may play an important role:

- **Gamma Ray Bursts (GRBs):** According to the collapsar model of GRBs [e.g. Woosley (1993)], a highly relativistic jet emerges from a massive star that ends its life in a core collapse SN. The jet should initially penetrate the mantle of the star, and the shock that decelerates the jet is expected to be ultra relativistic and radiation mediated.

- **Trans Relativistic SNe:** Several recent SN events, that were identified in very early stages of the explosion, have been shown to have a large fraction of the total energy stored in a mildly relativistic ($\gamma\beta \gtrsim 1$) ejecta. The existence of the mildly relativistic component suggests that a mildly relativistic RMS shock traversed the outer envelope of the progenitor star.

Understanding the structure and radiation spectrum of RRMS is important for understanding the nature of these events.

Several new effects come into play at mildly relativistic and relativistic RMS, which complicate the equations of the shock structure considerably, and inhibited a self consistent solution to date. These include 1. a non isotropic distribution of photons which inhibits the use of the diffusion approximation. 2. $e^+e^-$ pair production 3. relativistic corrections to the radiation emission mechanisms. Studying the shock structure in this range requires a realistic description of the transport of radiation, which takes into account the relativistic effects.

An over simplified solution of the structure of a relativistic RMS, neglecting pair production and taking into account photon scattering only, while neglecting photon production and relativistic corrections, was presented in (Levinson & Bromberg 2008). This solution may be applicable only
in a case where the UpStream (US) material holds a significant photon component which keeps the flow at low temperatures throughout the shock.

In part § III of this work we numerically solve the steady state structure of RRMS in the shock frame, in which the upstream consists of a cold plasma of protons and electrons, for upstream Lorentz factors $\leq 30$ and densities $\ll 10^{25}$ cm$^{-3}$. The solution is obtained by a simultaneous integration of the energy, momentum and particle conservation equations and the equation of radiation transport. The electrons, positrons and protons are described as a fluid with a space dependent velocity. The positrons and electrons are assumed to have a Maxwellian distribution with a characteristic temperature in the fluid rest frame while the protons are assumed to be cold. The Radiation mechanisms that are taken into account in the analysis include Compton scattering, pair production and annihilation and Bremsstrahlung emission and absorption. Other radiation mechanisms, e.g. double Compton scattering, are argued to have a minor effect on the results.

3. SN Shock Breakout

It has long been suggested that a strong outburst of thermal Ultraviolet or X-ray radiation may be observable when the RMS reaches the edge of the star, with a characteristic temperature $\sim 0.1$ keV. (e.g. Colgate 1974; Falk 1978; Klein & Chevalier 1978; Ensman & Burrows 1992; Matzner & McKee 1999; Blinnikov et al. 2000). These outbursts carry important information about the progenitor star (Arnett 1977) including a direct measure of the star’s radius. During the past three years, the wide field X-ray detectors on board the Swift satellite detected luminous X-ray outbursts preceding a SN explosion in two cases, SN2006aj and SN2008D (Campana et al. 2006; Soderberg et al. 2008). These bursts show an unexpectedly nonthermal spectrum extending to high energies, $\geq 10$ keV, which is inconsistent with the thermal radiation in the RMS DS.

In this work we follow the dynamics of the shock as it approaches the edge of a star and produces the radiation. In particular, using the analytic estimates of a RMS structure found in § III, we show that the expected emission from such a breakout reaches energies $\geq 10$ keV, if the shock velocity is larger than 0.1$c$, while the DS temperature is $\sim 0.1$ keV. This analysis has an impact on the interpretation of past and future breakout events, and may also assist in the design of future observational and instrumental efforts to explore SN breakouts.

4. Organization of the thesis

In part § II we discuss the possibility that CRs between the knee and the ankle are produced in shocks of TRSNe. In part § III we discuss the physics of RMS, and present numerical and
analytical results for the structure of RMS. We then discuss an application of the results to SN shock breakout. In part § IV we discuss the results presented in this thesis.

Part II

Cosmic Rays from TRSNe

We show that Galactic TRSNe may be the sources of \( \sim 10^{15} \) eV to \( \sim 10^{18} \) eV CRs. We first derive in § 5 constraints, which apply to both SNe and TRSNe, that must be satisfied by Galactic sources of \( \sim 10^{15} \) eV to \( \sim 10^{18} \) eV. We then show in § 6 that while these constraints are not satisfied by SNe, they may be satisfied by TRSNe. Our results are summarized and their implications are discussed in § 14.

5. Constraints on Galactic sources of CRs

We consider three constraints which must be satisfied by the candidate sources of CRs at high energy, \( \varepsilon \sim 10^{18} \) eV. First we consider in § 5.1 constraints on the rate of occurrence of the sources in the Galaxy and on their energetics. The constraints derived in § 5.1 are applicable to any (Galactic) candidate source. Next, we consider in § 5.2 and § 5.3 constraints that are applicable to SNe and TRSNe. These sources are expected to accelerate particles to high energies through the collisionless shocks which they drive into the plasma surrounding the exploding star. In § 5.2 we derive constraints on the velocity of the shell ejected by the explosion, and in § 5.3 we derive constraints on the structure of the ejecta.

5.1. Rates and energetics

In order for a certain type of sources to be the main contributor to the observed CR flux at some energy \( \varepsilon_0 \), the sources must satisfy the following conditions:

1. The energy production rate at \( \varepsilon_0 \) should be sufficient to explain the observed flux, and

2. The Galactic event rate, that is the rate of occurrence of the sources in the Galaxy, should not be much lower than one event per Galactic confinement time of CRs with energy \( \varepsilon_0 \).

Consider a source with a Galactic occurrence rate of \( \dot{N}_s \) and an energy per event of \( E_s \). \( E_s \) denotes the energy available for the acceleration of \( 10^{18} \) eV CRs, rather than the total energy
released. In SN explosions, for example, the energy available for CR acceleration is the energy deposited in the shock wave driven by the ejecta. Since the maximum energy to which particles may be accelerated increases with shock velocity (see § 5.2), the energy available for acceleration of particles to $10^{18}$ eV is the energy carried by the fastest (outermost) part of the ejecta, which drives, at early times, a sufficiently fast shock. This energy may be significantly smaller than the total kinetic energy of the (slower) ejecta. We assume that the source produces an energy spectrum of accelerated particles given by

$$\varepsilon^2 \frac{dN}{d\varepsilon} = \zeta_s(\varepsilon) E_s.$$  \hspace{1cm} (1)

Here $\zeta_s(\varepsilon)$ is the fraction of $E_s$ that is deposited in CRs within a logarithmic energy interval around $\varepsilon$. The resulting energy density of CRs per unit logarithmic particle energy is

$$\varepsilon^2 n(\varepsilon) = \frac{\zeta_s(\varepsilon) \tau_c(\varepsilon) E_s}{V_G},$$  \hspace{1cm} (2)

where $n(\varepsilon)$ is the number density per unit CR energy, $\tau_c$ is the confinement time of the particles in the Galaxy, and $V_G$ is the effective volume of the Galaxy in which the particles are confined. We assume that this effective volume is energy independent.

To reduce uncertainties we compare the resulting densities with those of low energy, $10^9$ eV, CRs, assuming that in this energy range the CRs are accelerated predominantly by SNe. Using Eq. (2), the ratio of CR number densities (per unit CR energy) at low and high energy, $n_l/h = n(\varepsilon = \varepsilon_{l,h})$, is

$$\frac{\varepsilon^2 n_h}{\varepsilon_l^2 n_l} \approx \frac{\dot{N}_s}{\dot{N}_{SN}} \frac{\tau_c(\varepsilon_h)}{\tau_c(\varepsilon_l)} \frac{E_s}{E_{SN}} \frac{\zeta_s(\varepsilon_h)}{\zeta_{SN}(\varepsilon_l)},$$  \hspace{1cm} (3)

where $E_{SN}$ is the characteristic kinetic energy of a SN explosion. For $\varepsilon_h = 10^{18}$ eV and $\varepsilon_l = 10^9$ eV the observed ratio is (e.g. Nagano & Watson 2000, and references within)

$$\frac{\varepsilon^2 n_h}{\varepsilon_l^2 n_l} \approx (10^{15}/10^9)^{-0.7} (10^{18}/10^{15})^{-1.0} \approx 10^{-7}.$$  \hspace{1cm} (4)

Using Eqs. (3) and (4), we find that the sources of CRs at $\varepsilon = \varepsilon_h = 10^{18}$ eV must satisfy

$$\frac{N_s}{N_{SN}} \approx 10^{-7} \zeta^{-1} \left( \frac{E_s}{E_{SN}} \right)^{-1},$$  \hspace{1cm} (5)

where $N_s = \dot{N}_s \tau_c(\varepsilon_h)$ and $N_{SN} = \dot{N}_{SN} \tau_c(\varepsilon_l)$ are the average numbers of events that contribute to the observed flux at any given time at energies $\varepsilon_h = 10^{18}$ eV and $\varepsilon_l = 10^9$ eV respectively, and $\zeta \equiv \zeta_s(10^{18}$ eV)/$\zeta_{SN}(10^9$ eV). Using the confinement time of 1 GeV protons in the Galaxy, $t_c(10^9$ eV) $\approx 10^{7.5}$ yr (Webber & Soutoul 1998; Yanasak et al. 2001), Eq. (5) may be written as

$$N_s \approx 0.03 \zeta^{-1} \left( \frac{E_s}{E_{SN}} \right)^{-1} \dot{N}_{SN,-2},$$  \hspace{1cm} (6)
where $\dot{N}_{\text{SN}} = 0.01 \tilde{N}_{\text{SN},-2} \text{ yr}^{-1}$. The requirement that the event rate should exceed one event per Galactic confinement time, $N_s \gtrsim 1$, implies

$$\frac{\zeta E_s}{E_{\text{SN}}} < 0.03 \tilde{N}_{\text{SN},-2}. \quad (7)$$

The confinement time of CRs of energy $\sim 10^{18} \text{eV}$ is likely to be significantly larger than the light crossing time of the Galaxy, $\sim 10^4 \text{yr}$, due to the following argument. The observed energy evolution of the depth of maximum, $X_{\text{max}}$, of extreme air showers suggest that composition of CRs at $\sim 10^{18} \text{eV}$ is dominated by nuclei with moderate atomic number $Z$ (Gaisser et al. 1993; Nagano & Watson 2000). Consider therefore the Larmor radius of a CR particle with atomic number $Z = 10 Z_1$ and an energy $\varepsilon = 10^{18} \varepsilon_{18} \text{eV},$

$$R_L \approx 40 B_{-5.5}^{-1} \varepsilon_{18}^{Z_1 - 1} \text{pc},$$

where $B = 3B_{-5.5} \mu \text{G}$ is the Galactic magnetic field. Since the Larmor radius of such particles is much smaller than the thickness of the CR disk, $\sim 1 \text{kpc}$, the propagation of $\varepsilon = 10^{18} \text{eV}$ CRs is expected to be strongly affected by the Galactic magnetic field. The transition to non-confined particles, for which the confinement time is comparable to the light crossing time, should then occur at $\gtrsim 10^{19} Z_1 \text{eV}$. Using therefore $\tau_c(10^{18} \text{eV}) = \tau_{4.5} 10^{4.5} \text{yr}$, Eqs. (3) and (4) give

$$\frac{\dot{N}_s}{\dot{N}_{\text{SN}}} = 10^{-4} \tau_{4.5}^{-1} \left( \frac{E_s}{E_{\text{SN}}} \right)^{-1} \zeta^{-1}. \quad (8)$$

The constraint that the Galactic event rate be larger than one event per confinement time can be written as:

$$\frac{\dot{N}_s}{\dot{N}_{\text{SN}}} > 3 \times 10^{-3} \tau_{4.5}^{-1} \tilde{N}_{\text{SN},-2}. \quad (9)$$

Eqs. (7), (8) and (9) (any one of these can be derived from the other two) describe the constraints on the occurrence rate and energetics of the sources of $\sim 10^{18} \text{eV}$ CRs.

A note is in order regarding the value of the parameter $\zeta \equiv \zeta_s(10^{18} \text{eV})/\zeta_{\text{SN}}(10^9) \text{eV}$. In the following, we consider sources that accelerate particles through collisionless shocks in the surrounding medium driven by fast ejecta. Note that although a complete theory, based on first principles, of particle acceleration in collisionless shocks is not available, there are strong theoretical arguments and observational evidence indicating that for both non-relativistic and relativistic shocks a considerable fraction of the post shock energy can be converted to relativistic particles with a spectrum that follows $\varepsilon^2 dN/d\varepsilon \approx \varepsilon^0$ (see, e.g. Blandford & Eichler 1987; Axford 1994; Waxman 2006, for reviews). If both SNe and the sources of $10^{18} \text{eV}$ CRs deposit a significant fraction of their energy in accelerated particles with a spectrum $dN/d\varepsilon \propto \varepsilon^{-2}$, the value of $\zeta$ would be of order unity.
5.2. Maximum energy

We assume that the particles are accelerated in the vicinity of a collisionless shock wave with (a time dependent) velocity \( v_S = \beta_S c \) that is driven into the surrounding medium by an ejecta of mass \( M_{ej} \), initial velocity \( v_{ej} = \beta_{ej} c \) and kinetic energy \( E_{ej} = (\gamma_{ej} - 1)M_{ej}c^2 \). Here \( \gamma_{ej} = (1 - \beta_{ej}^2)^{1/2} \) is the ejecta’s Lorenz factor. We consider two options for the density distribution of the surrounding medium: A homogeneous inter-stellar medium (ISM) or a progenitor wind with a density profile \( \rho \propto r^{-2} \).

At any given time, the maximum energy to which particles can be accelerated is given by

\[
\varepsilon_{\text{max}} \sim Zc\beta_S B_d r_S
\]

(e.g. Waxman 2004c, for a recent review). Here, \( r_S \) is the radius of the shock and \( B_d \) is the minimum of the magnetic field in the pre-shock (upstream) and post-shock (downstream) plasma, measured in downstream frame. We should stress that the energy given by Eq. (10) is an upper limit, and that the maximum energies attained are likely to be lower. For example, if particles are accelerated via Fermi’s diffusive shock acceleration, the maximal energy would be a few times lower for Bohm limit diffusion, and much lower for slower diffusion.

There is growing evidence that the magnetic field in the vicinity of non-relativistic collisionless shocks in SNRs and gamma-ray bursts (GRBs) afterglows is amplified to values greatly exceeding the ambient magnetic field both in the upstream and downstream. In the past few years, high resolution X-ray observations have provided indirect evidence for magnetic fields of values as high as 100 \( \mu \)G in the \( (v_S \sim \text{few } \times 1000 \text{ km s}^{-1}) \) shocks of young SNRs (see Bamba et al. 2003; Vink & Laming 2003; Völk et al. 2005). For relativistic shocks in the afterglow phase of GRBs, a very large magnetic field in the downstream is inferred from measurements (see Waxman 2006, for review), that can not be explained as a mere compression of the ambient field and requires a fair fraction of the kinetic energy of the shock to be transformed into magnetic energy. Li & Waxman (2006) have furthermore shown that in GRB afterglow shocks strong amplifications of the magnetic field in the upstream can be also inferred. We therefore assume that the magnetic field is amplified by the shock both in the upstream and in the downstream to values close to equipartition

\[
B_d = \sqrt{\epsilon_B 8\pi \rho v_S^2 \gamma_S^2}.
\]

Initially, the shock and ejecta velocities (which are similar) are constant with time while the radius increases linearly. For the density distributions considered here the maximal acceleration energy is attained when the ejecta begins to decelerate. This happens roughly when the energy in the swept up material is comparable to the initial ejecta energy, i.e. when \( M_{\text{swept}} \sim M_{ej}/\gamma_{ej} \).
In case the shock is propagating into a homogeneous ISM, the deceleration radius is given by

\[ r_{\text{hom}} \approx \left( \frac{M_{\text{ej}}}{\frac{4\pi}{3} n_0 m_p \gamma_{\text{ej}}} \right)^{1/3}, \tag{12} \]

where \( n_0 \) is the ISM particle density per cm\(^3\). Using Eqs. (10), (11) and (12), the maximum energy of an accelerated particle is

\[ \varepsilon < 3 \times 10^{17} Z \epsilon_{B,-1}^{1/2} \left( \frac{M_{\text{ej}}}{10 M_\odot} \right)^{1/3} n_0^{1/6} \beta_{\text{ej}}^{2/3} \gamma_{\text{ej}}^{2/3} \text{eV}. \tag{13} \]

Here \( \epsilon_B = 0.1 \epsilon_{B,-1} \) and \( \beta_{\text{ej}} = 10^{-2} \beta_{\text{ej},-2} \). For a shock propagating into a progenitor wind with a density profile \( \rho = \dot{M}/4\pi r^2 \nu_w \), where \( \dot{M} \) is the mass loss rate and \( \nu_w \) is the wind velocity, the deceleration radius is

\[ r_{\text{wind}} \approx \frac{M_{\text{ej}} \nu_w}{\gamma_{\text{ej}} \dot{M}}. \tag{14} \]

Using Eqs. (10), (11) and (14), the maximum energy of an accelerated particle is

\[ \varepsilon < 10^{16} Z \epsilon_{B,-1}^{1/2} \left( \frac{\dot{M}_{-5}}{\nu_{w,8}} \right)^{1/2} \beta_{\text{ej},-2}^{2/3} \gamma_{\text{ej}} \text{eV}. \tag{15} \]

Here \( \dot{M} = 10^{-5} \dot{M}_{-5} M_\odot/\text{yr} \) and \( \nu_w = 10^8 \nu_{w,8} \text{cm/s} \). The maximum energy is independent in this case of the ejecta mass, i.e. of the deceleration radius, since for a \( 1/r^2 \) density profile the maximum energy (for a fixed shock velocity) is independent of the shock radius.

Note, that this estimate is valid provided shock deceleration occurs while it is propagating within the \( \rho \propto r^{-2} \) wind profile. The \( \rho \propto r^{-2} \) dependence of wind density extends up to a radius where \( \rho_w \nu_w^2 \approx P_{\text{ISM}} \), and the wind mass contained within this radius is

\[ M_{\text{w}} \approx \frac{\dot{M}_{-5}^{3/2}}{(4\pi \nu_w P_{\text{ISM}})^{1/2}} \approx 0.3 M_\odot \frac{\dot{M}_{-5}^{3/2}}{\nu_{w,8}^{1/2} P_{\text{ISM},-0.5}^{1/2}}. \tag{16} \]

where the ISM pressure is \( P_{\text{ISM}} = 0.3 \text{eV/cm}^3 P_{\text{ISM},-0.5} \). The maximum energy is given by eq. (15) for \( M_{\text{ej}}/\gamma_{\text{ej}} < M_{\text{w}} \), and it is lower otherwise.

### 5.3. Energy distribution within the ejecta

SN explosions produce non-uniform ejecta, with velocity rising towards the front edge of the ejecta. The shock driven by the fastest, outermost part of the ejecta is capable of accelerating particles to the highest energy. Here we consider the case where a small and fast part of the ejecta is responsible for accelerating cosmic rays to energies \( \varepsilon_h \sim 10^{18} \text{eV} \), and constrain the
energy distribution of the slower parts of the ejecta by requiring that they do not produce a flux of lower energy CRs that exceeds the observed one.

We denote the amount of energy in ejecta moving faster than $\beta c$ by $E_k(>\beta)$. Since we consider in this section a limited range of particle energies, we assume that each part of the ejecta contributes a constant fraction $\zeta_{ej}$ to each logarithmic bin of particle energies at $\varepsilon < \varepsilon_{\text{max}}(\beta)$ (ignoring the possible energy dependence of $\zeta_{ej}$). Under this assumption, the resulting energy flux per logarithmic particle energy interval is $\varepsilon^2 n(\varepsilon) \propto \tau_c(\varepsilon) E_k(>\beta(\varepsilon))$. Here $\beta(\varepsilon)$ ($\propto \varepsilon^{1/2}$ for $\beta \ll 1$) is the shock velocity required for accelerating up to $\varepsilon$, i.e. $\varepsilon_{\text{max}}(\beta) = \varepsilon$.

Assuming that the part of the ejecta with velocities larger than some $\beta_h$ is responsible for the CR spectrum at some $\varepsilon_h$, requiring the slower parts of the ejecta do not produce a flux of cosmic rays at lower energy that exceeds the observed flux implies

$$\frac{E_k(>\beta(\varepsilon))}{E_k(>\beta(\varepsilon_h))} \leq \frac{\varepsilon^2 n_{\text{obs}}(\varepsilon)}{\varepsilon^2 n_{\text{obs}}(\varepsilon_h)} \left[ \frac{\tau_c(\varepsilon)}{\tau_c(\varepsilon_h)} \right]^{-1},$$

(17)

where $n_{\text{obs}}$ stands for the observed CR number density (per unit CR energy). Since the measured energy flux per logarithmic particle energy interval in the range $10^{15}\text{eV} < \varepsilon < 10^{18}\text{eV}$ is roughly proportional to $E^{-1}$, and the confinement time is expected to decrease with energy, the energy distribution must satisfy

$$\frac{E_k(>\beta(\varepsilon))}{E_k(>\beta(\varepsilon_h))} < \left( \frac{\varepsilon}{\varepsilon_h} \right)^{-1}.$$ 

(18)

6. Application to SNe and TRSNe

6.1. SNe

Eqs. (13) and (15) imply that typical SNe driven shocks, characterized by $\beta_{ej} \sim 10^{-2}$, are probably too slow to accelerate particles to $10^{18}\text{eV}$, especially considering that these are optimistic upper limits. Another difficulty in attributing the production of $10^{18}\text{eV}$ CRs to ordinary SNe is related to the constraint of Eq. (8). According to this constraint, if SNe were capable of accelerating particles to energies as high as $10^{18}\text{eV}$, the flux produced at these energies would exceed the observed flux by a large factor unless $\zeta \sim 10^{-4}$, i.e. unless the energy carried by $10^{18} \text{eV}$ CRs accelerated by the SN shock is smaller by a factor of $10^4$ than that carried by $10^9 \text{eV}$ CRs accelerated by the same shock. Such a strong suppression is not expected for the case when the maximum acceleration energy exceeds $10^{18} \text{eV}$, since in this case we expect equal energy to be deposited per logarithmic CR energy interval (see discussion at the end of section §5.1). For example, attributing this factor to an acceleration spectrum $dn/d\varepsilon \propto \varepsilon^{-p}$ with $p$ different from 2, would require $p > 2.4$ which seems inconsistent with theory and SNR radio observations.

One may argue that the above problem may be circumvented by assuming that $10^{18}\text{eV}$ CRs are
produced only by the fast, $\beta \gg 10^{-2}$, component of the SN ejecta, which carries only a small fraction of the total kinetic energy. This would lead, however, to violation of the constraint of Eq. (18). The typical energy distribution in SN ejecta is $E_k(\beta) \propto \beta^{-5}$ for BSG progenitors and $E_k(\beta) \propto \beta^{-6}$ for RSG progenitors (Matzner & McKee 1999). At non relativistic velocities $\epsilon_{\text{max}} \propto \beta^2$, implying $E_k(\beta(\epsilon)) \propto \epsilon^{-5/2}$ or $E_k(\beta(\epsilon)) \propto \epsilon^{-3}$ for BSGs and RSGs respectively. This is in clear contradiction with Eq. (18). That is, if one assumes that the fast part of the ejecta produces the observed $10^{18}$eV CR flux, then the predicted flux at lower energies would far exceed the observed flux.

Bell & Lucek (2001) have argued that SN driven shocks are candidate sources of CRs above $10^{18}$eV, based on an estimate of the maximum acceleration energy. They have considered a shock of velocity of 40,000km/s driven into a massive wind, characterized by $\dot{M} \sim 1$ and $v_{w,8} \sim 10^{-2}$, as inferred from radio observations of SN1993J. Indeed, for such parameters the SN driven shock may allow acceleration to $> 10^{18}$eV, as indicated by Eq. (15). However, it should be realized that SN1993J is still young, and that the measured velocity therefore represents only the fast front edge of the ejecta, which contains only a small fraction of the ejecta mass and energy. This is well illustrated in the modeling of this SN by Woosley et al. (1994), where only a small fraction of the total mass and energy lies in such a high velocity component of the ejecta (see e.g. their fig. 6). As explained above, if the tail of the ejecta velocity distribution can account for the cosmic ray flux at some energy $\epsilon_h \sim 10^{18}$ eV, then the steep profile of energy as a function of velocity would imply a CR flux at lower energies that far exceeds the observed flux.

6.2. TRSNe

During the past decade, four GRBs were observed to be associated with core collapse SNe of type Ic: GRB980425/SN1998bw (Galama et al. 1998), GRB030329/SN2003dh (Hjorth et al. 2003; Stanek et al. 2003), GRB031203/SN2003lw (Tagliaferri et al. 2004), and XRF060218/SN2006aj (Campana et al. 2006). All but GRB030329 were very faint, 4-5 orders of magnitude fainter than an average cosmological GRB. In all 4 cases there is evidence from radio observations for deposition of a significant part of the kinetic energy in a mildly relativistic part of the ejecta (Soderberg et al. 2006, and references therein). Although radio observations alone do not allow one to uniquely determine the energy carried by the relativistic ejecta (Waxman & Loeb 1999; Waxman 2004a), additional observational information enabled such a unique determination in two cases. For GRB980425/SN1998bw there is a robust estimate of the energy deposited in the fast, $\beta \geq 0.85$, part of the ejecta, based on long term radio and X-ray observations, which yield $E(\beta \geq 0.85) = 10^{49.7}$ erg (Waxman 2004b). Similar values are inferred for XRF060218/SN2006aj from prompt and afterglow X-ray observations (Waxman et al. 2007).
The SN light curves of SN1998bw, SN2003dh, and SN2003lw appear to be qualitatively similar, implying SN ejecta mass and energy of \( E_{ej} \sim 5 \times 10^{52} \text{ erg} \), \( M_{ej} \sim 10 M_\odot \) (e.g. table 6 of Mazzali et al. 2006). Note, that the energy may be lower, \( \sim 1 \times 10^{52} \text{ erg} \), if the explosion is not spherical, which is likely (Maeda et al. 2006). The SN light curve of SN2006aj is qualitatively different than the others, implying \( E_{ej} \sim 2 \times 10^{51} \text{ erg} \) and \( M_{ej} \sim 2 M_\odot \) (in this case, the explosion is probably not highly non spherical; Mazzali et al. 2006). The inferred \( E_{ej} \) and \( E(\beta \geq 0.85) \) imply that a significant fraction of the kinetic energy is deposited in a fast, mildly relativistic part of the ejecta, \( E(\beta \geq 0.85)/E_{ej} \gtrsim 10^{-2} \).

Two points should be emphasized here. First, GRBs associated with TRSNe are different than cosmological GRBs: The amount of energy carried by relativistic ejecta in TRSN explosions, \( \sim 10^{49.5} \text{ erg} \), is much lower than that of cosmological GRBs, which is \( \sim 10^{51} \text{ erg} \), and the fastest part of the ejecta in a TRSN is only mildly relativistic, unlike the highly relativistic ejecta inferred from observations of GRBs. Second, the mechanism responsible for depositing \( \sim 10^{49.5} \text{ erg} \) in the mildly relativistic part of the ejecta is not understood. For the inferred values of \( E_{ej}/M_{ej}c^2 \sim 10^{-3} \), the acceleration of the SN shock near the edge of the star is expected to deposit only \( \sim 10^{-7}E_{ej} \) in the part of the ejecta expanding with \( \gamma \beta > 1 \) (Tan et al. 2001). This suggests that the mildly relativistic component is driven not (only) by the spherical SN shock propagating through the envelope, but possibly by a more relativistic component of the explosion, e.g. a relativistic jet propagating through the star (Aloy et al. 2000; Zhang et al. 2003). For the present analysis, we use therefore only the observational constraints on the energy distribution within the ejecta.

With only 4 detected events, the TRSNe rate is rather uncertain. Estimates of the local \((z = 0)\) rate range from \( \sim 10^2 \) to \( \sim 10^4 \text{Gpc}^{-3} \text{yr}^{-1} \) (Guetta & Della Valle 2007; Soderberg et al. 2006; Liang et al. 2006; Pian et al. 2006). Comparing this to the local SN rate, \( \approx 10^6\text{Gpc}^{-3} \text{yr}^{-1} \) (e.g. Cappellaro et al. 1999), we find \( \dot{N}_{\text{TRSN}}/\dot{N}_{\text{SN}} \approx 10^{-2.5 \pm 0.5} \). This estimate is consistent with the estimate of Podsiadlowski et al. (2004) of the rate of ”hypernovae” in galaxies similar to the Milky Way. As mentioned in the introduction, the TRSNe rate may be higher if some TRSNe are not associated with GRBs (and hence were not identified following a GRB trigger). With these estimates at hand, we are now ready to determine whether or not TRSNe meet the constraints derived in \S 5.

First, for \( \beta \approx 0.8 \) Eqs. (13) and (15) imply that the mildly relativistic part of TRSNe ejecta may indeed accelerate CRs up to energy exceeding \( 10^{18} \text{eV} \). Second, for an energy \( E_r \sim 10^{49.5} \text{ erg} \) in the relativistic part of the ejecta we have \( E_r/E_{SN} \sim 10^{-1.5} \), satisfying the energetics constraint, Eq. (7), and \( \dot{N}_{\text{TRSN}}/\dot{N}_{\text{SN}} \approx 10^{-2.5 \pm 0.5} \), satisfying the rate constraint, Eq. (8). Note, that for TRSNe we expect \( \zeta \sim 1 \), based on their radio and X-ray observations. These observations imply that the collisionless shocks driven by the mildly relativistic ejecta into the wind surrounding the progenitor transfer a significant fraction of the energy to a population of shock accelerated
electrons, and that the accelerated electron spectrum follows $\varepsilon^2 dN/d\varepsilon \approx \varepsilon^0$ over a wide range of energies, from electrons emitting synchrotron radiation in the radio band over at least $\sim 5$ decades of energy to those emitting radiation in the X-ray band (Waxman 2004b). Thus, if nuclei are accelerated with similar efficiency to a similar energy power spectrum, the fraction of energy deposited in $\sim 10^{18}$ eV CRs would be similar to that deposited by SNe in CRs accelerated to $\sim 10^9$ eV.

Let us next consider the constraint of Eq. (17) on the energy distribution within the ejecta. Consider a TRSN with a total kinetic energy of $E_{ej} = 10^{52}$ erg, an ejecta mass $M_{ej} = 10M_{ej,1}M_\odot$ and an energy $E_r$ in a mildly relativistic, $\beta \geq 0.8$, part of the ejecta. Using Eq. (15), the ratio of the maximum energies of particles accelerated by the relativistic part of the ejecta, $\varepsilon_{\text{max},r}$, and by the bulk of the ejecta, $\varepsilon_{\text{max},ej}$, is

$$\varepsilon_{\text{max},r}/\varepsilon_{\text{max},ej} \approx 10^3 E_{ej,52}^{-1} M_{ej,1},$$

and the constraint of Eq. (17) is satisfied in the energy range of $10^{15}$ eV $- 10^{19}$ eV as long as

$$E_r > 10^{49} E_{ej,52}^2 \tau_c(\varepsilon_{\text{max},r}) \tau_c(\varepsilon_{\text{max},ej}) \text{erg.}$$

Assuming that $\tau_c \propto \varepsilon^{\delta}$ we have

$$E_r > 10^{49} E_{ej,52}^{2-\delta} M_{ej,1}^{1-\delta} 10^{34} \text{erg.}$$

An upper limit to the value of $\delta$ in the relevant energy range, $10^{15}$ eV $- 10^{19}$ eV, may be obtained as follows. First note that the confinement time of $Z = 10$ particles at $\varepsilon = 10^9 Z$ eV is $10^{7.5}$ yr while and at $\varepsilon = 10^{17} Z$ eV it is larger than $10^{4.5}$ yr, implying that the average value of $\delta$ between these energies satisfies

$$\bar{\delta} = \frac{\log(\varepsilon_{hi}/\varepsilon_{li})}{\log[\tau_c(\varepsilon_{hi})/\tau_c(\varepsilon_{li})]} \leq 0.4.$$ 

In fact, the value of $\delta$ in the energy range of $10^{15}$ eV $- 10^{19}$ eV must be lower than this average. The grammage (column density) traversed by CRs of energy $< 1$ TeV before they escape our Galaxy is $\Sigma_{\text{conf}} \approx 9(E/10 Z$ GeV)$^{-\delta} g$ cm$^{-2}$ with $\delta \approx 0.6$ (Engelmann et al. 1990; Stephens & Streitmatter 1998; Webber et al. 2003), suggesting $\tau_c \propto 10^{7.5}(E/Z$ GeV)$^{-0.6}$ yr for $E/Z < 10^{14}$ eV. As the value of $\delta$ at energies $10^9 Z$ eV $- 10^{12} Z$ eV is higher than the average, $\delta \leq 0.4$, the value of $\delta$ must be lower at energies $10^{13} Z$ eV $- 10^{17} Z$ eV. It is reasonable to assume therefore $10^{38} < 10$, which implies that the constraint of Eq. (21) may be satisfied for reasonable values of the parameters.

Finally, we note that Milgrom & Usov (1996) have suggested that cosmological GRBs, which occur in our galaxy once every $\sim 10^5$ years, could be the sources of the CRs in the energy range of $10^{14}$ eV to $10^{19}$ eV. This suggestion is quite distinct from the one presented in this article. While Milgrom & Usov (1996) considered acceleration by the highly relativistic jets
of cosmological GRBs, where $\sim 10^{51}$ erg is carried by a highly relativistic, $\gamma \sim 10^{2.5}$ ejecta, we consider acceleration by mildly relativistic, $\gamma/\beta \sim 1$, less energetic, $\sim 10^{49.5}$ erg, and more frequent TRSN explosions.
Part III

Radiation Mediated Shocks

In this part we study, analytically and numerically, the structure of RMS, both NR and relativistic. In section §7 we review the physics of RMS, analyze qualitatively the shock structure, and motivate the main assumptions. In section §8 we write down the conservation and transport equations that are numerically solved, in physical and dimensionless forms. In section §9 we present the numerical iteration scheme used to obtain the solutions and apply it to several test cases. In section §10 we present the numerical solutions of the shock structure and spectrum. In section §11 we give some simple analytic estimates for the structure of the shock, which reproduce the main results of the numerical calculations. In section §12 we give, for completeness, results of a detailed numerical solution of a non relativistic RMS, and compare it with previously known results. In §13 we discuss the implications of the analysis presented in this chapter to the expected properties of SN X-ray outbursts. We show that breakout velocities \( \beta_u > 0.1 \) may be reached in the explosions of Blue Super Giant (BSG) and Wolf-Rayet (WR) stars (§13.2), and argue that the X-ray outbursts accompanying breakouts from such stars may include a hard component with photon energies reaching tens or even hundreds of keV (§13.3).

Throughout this chapter the subscripts \( u, d \) refer to the US and DS values, respectively. \( n \) stands for number density of a species of particles, and if not mentioned otherwise refers to protons. A summary of the notations repeatedly used in this chapter appears in appendix §A.

7. The physics of RMS

Consider a steady state shock traveling with velocity \( c\beta_u \) through an infinitely thick, cold plasma of protons and electrons, with US rest frame density \( n_u \). The thermal and radiation pressures in the asymptotic far DS, which are determined by conservation laws and thermal equilibrium, are given by \( 2n_d T_d \) and \( a_{BB} T_d^4 / 3 \) respectively, where \( n_d \) and \( T_d \) are the far DS proton density and temperature, and \( a_{BB} = \pi^2 / 15(\hbar c)^{-3} \) is the Stefan-Boltzmann energy density coefficient. The radiation pressure grows much faster than the thermal pressure as a function of \( \beta_u \), and at high enough \( \beta_u \),

\[
\beta \gg \left( \frac{n}{a_{BB}} \right)^{1/6} (m_p c^2)^{-1/2} \sim 3 \times 10^{-4} \left( \frac{n}{10^{20} \, \text{cm}^{-3}} \right)^{1/6},
\]

the DS pressure is dominated by the radiation. To obtain Eq. (22), note that at low shock velocities, where the radiation pressure is negligible, \( T_d \sim \varepsilon \) and \( n_d \approx 4n_u \), where \( \varepsilon \approx \beta_u^2 m_p c^2 / 2 \) is the kinetic energy per proton in the US. In this section we discuss the key physical characteristics
of such shocks. The conditions under which RMSs exist are given in § 7.7.

7.1. Far DS conditions

In the far DS, which is in thermal equilibrium, the conditions are completely determined by conservation of energy, momentum and particle fluxes,

\[ n_d \Gamma_d \beta_d = n_u \Gamma_u \beta_u, \]
\[ 4 \Gamma_u^2 \beta_d p_{\gamma,d} = \Gamma_u \beta_u (\Gamma_u - \Gamma_d) n_u m_p c^2, \]
\[ (4 \Gamma_d^2 \beta_d^2 + 1) p_{\gamma,d} = \Gamma_u \beta_u (\Gamma_u \beta_u - \Gamma_d \beta_d) n_u m_p c^2, \]

where \( p_{\gamma,d} = 1/3 a_{BB} T_d^4 \) is the far DS radiation pressure, and where the plasma pressure in the DS was neglected. Eqs. (23) can be solved for \( \beta_d \) and \( T_d \). In the NR and ultra relativistic limits the solution reduces to the expressions

\[ T_d \approx \left( \frac{21 n_u \beta_u^2 m_p c^2}{8 a_{BB}} \right)^{1/4} \approx 0.41 (n_u,15)^{1/4} \beta_u^{1/2} \text{ keV (NR)}, \]
\[ \beta_d \approx \beta_u / 7, \]

and

\[ T_d \approx \left( \frac{2 \Gamma_u^2 n_u m_p c^2}{a_{BB}} \right)^{1/4} \approx 0.385 \Gamma_u^{1/2} n_u,15^{1/4} \text{ KeV (Relativistic)}, \]
\[ \beta_d \approx 1/3, \]

respectively, where \( n_u = 10^{15} n_u,15 \text{ cm}^{-3} \). The condition for a radiation dominated DS, Eq. (22), can be obtained by comparing \( T_d \) with \( \varepsilon_u \).

7.2. From DS radiation domination to radiation mediation

For shocks satisfying Eq. (22), the far DS pressure is dominated by radiation. It is not a priory trivial that the pressure is dominated by radiation in the velocity transition region of such shocks since photons that are generated in the DS are able to diffuse upstream over a finite distance only. We next illustrate that under a wide range of conditions, the radiation does indeed dominate the pressure in the velocity transition region.

Consider a hypothetical shock, in which the velocity transition is mediated by some mechanism other than radiation. In the absence of radiation, the temperature immediately behind the velocity transition would be \( T \sim \varepsilon = (\Gamma_u - 1) m_p c^2 \). Photons generated in this region can diffuse upstream to a characteristic distance of \( L_{\text{diff}} > (\varepsilon n_e \sigma_T)^{-1} \), where \( c/\beta \) is the flow velocity, which is always non relativistic and values of \( L_{\text{diff}} \) larger than \( (\varepsilon n_e \sigma_T)^{-1} \) may be obtained due to
Klein Nishina (KN) corrections. For $\varepsilon \lesssim m_e c^2$, where pair production can be neglected, the energy density in photons that are produced by Bremsstrahlung can be estimated,

$$\frac{c_\gamma}{c_{th}} \approx \frac{Q_{Br} T L_{diff}}{\beta e n_p \varepsilon} > \frac{\alpha_e}{\beta^2} \sqrt{\frac{m_e c^2}{\varepsilon}} \approx 8 \alpha_e \frac{m_p}{m_e} \left( \frac{m_e c^2}{\varepsilon} \right)^{3/2} \gg 1,$$

where $Q_{Br}$ is the photon production rate by Bremsstrahlung at energy $\sim T$ and we used $\beta = \beta_u/4$, appropriate for NR shocks which are not radiation mediated.

For $\varepsilon > m_e c^2$, $e^+ e^-$ pair production will occur, enhancing the Bremsstrahlung emission, which in turn will increase the pair production. This unstable process is suppressed only when the radiation pressure becomes a significant fraction of the total pressure, resulting in a reduced temperature.

This implies that a shock with a radiation dominated DS and negligible radiation in the velocity transition region cannot exist if the transition region is smaller than $L_{diff}$. Once the pressure is dominated by photons, they will also mediate the shock.

### 7.3. The shock transition width

Physical quantities approach their far DS equilibrium values on length scales, which may vary by orders of magnitude for different quantities. In particular, as explained below, in RMSs the transition width of the velocity is determined by Compton scattering, and occurs on length scales which may be much smaller than the temperature transition width, which is determined by photon production (e.g. Bremsstrahlung).

#### 7.3.1. Velocity transition

For NR RMS the width of the velocity transition region is comparable to the distance $L_{diff} \sim (\beta_u n_c \sigma_T)^{-1}$ over which a photon can diffuse against the flow before being advected with the flow. To see that the velocity transition width can not be larger, note that once a proton reaches a point in the shock where the energy is dominated by photons, it experiences an effective force

$$\beta_u \frac{d\beta}{dx} m_p c^2 \sim \sigma_T \beta_u c_\gamma, \quad \sigma_T n_u \beta_u^3 m_p c^2,$$

implying a deceleration length of

$$L_{dec} \equiv \beta_u \left( \frac{d\beta}{dx} \right)^{-1} \sim \frac{1}{\sigma_T n_u \beta_u}.$$
This line of arguments can not be directly extended to relativistic shocks since KN corrections to the Compton scattering cross section depend on the a priori unknown photon frequency and plasma temperature which vary throughout the transition region. Also, as was shown numerically by Weaver (1976) and explained by Katz et al. (2009), at even non relativistic upstream energies of about 100 MeV, the temperature reached by the plasma during deceleration becomes a considerable fraction of $m_e c^2$, making single scattering energy shift and pair production important. The production of pairs also changes the simple estimate given above, since it changes both the scatterers number density and the optical depth for a photon crossing the shock.

### 7.3.2. Thermalization length

The region of the shock profile over which the temperature changes before it reaches $T_d$ can be extended to distances that are much larger than $L_{diff}$. To see this, consider the length scale that is required to generate the density of photons of energy $\sim T_d$ in the downstream, determined by thermal equilibrium, $n_{\gamma, eq} \approx p_{\gamma, d}/T_d$,

$$L_T \sim \beta u \frac{n_{\gamma, eq}}{Q_{\gamma, eff}}.$$  

(29)

where $Q_{\gamma, eff}$ is the effective generation rate of photons of energy $3T_d$. We use here the term "effective generation rate" due to the following important point. Photons that are produced at energies $\ll T_d$ may still be counted as contributing to the production of photons at $T_d$, since they may be upscattered by inverse-Compton collisions with the hot electrons to energy $\sim T_d$ on a time scale shorter than that of the passage of the flow through the thermalization length, $L_T/\beta u c$. The Bremsstrahlung effective photon generation rate is given by

$$Q_{\gamma, eff} = \alpha_e n_p n_e \sigma_T c \sqrt{\frac{m_e c^2}{T}} \Lambda_{eff} g_{eff},$$

(30)

where $g_{eff}$ is the Gaunt factor, $\Lambda_{eff} \sim \log\left[T/(h\nu_{min})\right]$ and $\nu_{min}$ being the lowest frequency of photons emitted by the plasma which manage to be upscattered to $3T_d$, which is dominated by Bremsstrahlung self absorption for far DS values.

The resulting thermalization length is

$$L_T \beta u n_u \sigma_T \sim \frac{1}{100 \alpha_e \Lambda_{eff} g_{eff} \sqrt{m_e c^2 T m_p c^2}} \varepsilon^2.$$  

(31)

This implies that for high shock velocities,

$$\beta u > 0.07 n_{15}^{1/30} (\Lambda_{eff} g_{eff})^{4/15},$$

(32)

the length required to produce the downstream photon density is much larger than the deceleration scale. For lower shock velocities, thermal equilibrium is approximately maintained throughout the shock.
7.4. Immediate DS and subsonic region in RRMS

When $L_T \gg L_{diff}$, the velocity transition of the shock ends without reaching the far DS equilibrium temperature. We refer to the part of the shock that lies $\beta_s^{-1}$ optical depths downstream of the velocity transition, as the “immediate DS”. In this region the photons mediating the shock are produced. To estimate the temperature of the immediate DS, $T_s$, we equate the number of photons produced by Bremsstrahlung and upscattered by inverse Compton to the required number of photons that carry the pressure at that point. The production rate, given by Eq. (30), combined with diffusion and conservation laws eventually leads to the estimate of the immediate DS temperature for NR RMS (Katz et al. 2009)

$$\beta_u = \frac{7}{\sqrt{3}} \left( \frac{1}{2} \alpha_e \Lambda_{\text{eff}} g_{\text{eff}} \right)^{1/4} \left( \frac{m_e}{m_p} \right)^{1/4} \left( \frac{T_s}{m_e c^2} \right)^{1/8} \approx 0.2 \Lambda_{\text{eff},1}^{1/4} \left( \frac{g_{\text{eff}}}{2} \right)^{1/4} \left( \frac{T_s}{10 \text{ keV}} \right)^{1/8},$$

where $\Lambda_{\text{eff},1} = 10 \Lambda_{\text{eff}}$. This result is also in agreement with Weaver (1976).

For RRMS we give a rough estimate of the average temperature in the first few optical depths of the immediate DS. The assumption we use is that the electron-positron pairs and the radiation are in Compton Pair Equilibrium (CPE). A thorough check of the self consistency of this assumption is given below in § 11.2. We equate the number of photons produced by Bremsstrahlung and by inverse Compton scattering of thermal pairs to the number of photons needed at the end of the shock, in a similar method to the one we used above for NR RMS.

Since pairs dominate the production, as the number density of pairs is much larger than that of protons, and neglecting Double Compton emission, the produced number of photons can be written as

$$\frac{n_\gamma}{n_l} = \frac{1}{3} \alpha_e \Lambda_{\text{eff}} g_{\text{eff,rel}}(\hat{T}) \beta_d^{-2},$$

where we wrote the free-free emission in the form

$$Q_{\gamma,\text{eff}} = \alpha_e \sigma_T c n_l^2 \Lambda_{\text{eff}} g_{\text{eff,rel}}(\hat{T}).$$

Here $g_{\text{eff,rel}}$ is the total Gaunt factor [defined by Eq. (35)] including all lepton-lepton Bremsstrahlung emission. For $10 < \Lambda_{\text{eff}} < 20$ and $60 \text{ keV} < T < m_e c^2$, the approximation

$$g_{\text{eff,rel}} \approx \Lambda_{\text{eff}} / 2$$

agrees with the results of Svensson (1984) to an accuracy of better than 25%. Substituting Eq. (36) in Eq. (34) we find

$$\frac{n_\gamma}{n_l} \approx 2.5 \left( \frac{\Lambda_{\text{eff}}}{15} \right)^2 (3 \beta_d)^{-2}.$$
Comparing equations (37) and (38), we see that if \( T \gtrsim 200 \text{ keV} \) there would be too many photons generated per lepton. At these high temperatures, the Compton \( y \) parameter is large and radiative Compton emission is negligible. We conclude that for relativistic shocks, \( T_s \approx 0.4 m_e c^2 \approx 200 \text{ keV} \).

Assuming that the radiation is in CPE with the plasma in the immediate DS we can estimate the speed of sound in matter (see appendix § C for details of the calculation of the speed of sound). It is found that the speed of sound is close to its ultra relativistic limit, \( \beta_{ss} = 1/\sqrt{3} \), and is quite robust due to a large number of pairs compared to protons. The velocity of the flow must quickly approach its DS value \( \beta_d < 1/3 \), which means that the flow would become subsonic, as opposed to the far US and the far DS which are both supersonic (\( \beta > \beta_{ss} \)).

### 7.5. Structure of RRMS

NR RMS and RRMS have a somewhat different structure, as we show in detail in this work. The main differences from the NR case are:

- The deceleration region is much larger than the naive estimate in terms of Thomson optical depths: The length grows with the upstream Lorentz factor \( \Gamma_u \) approximately as \( \Gamma_u^2 \).
- The mechanisms driving the deceleration of the plasma are both Compton scattering and pair production.
- The immediate DS of RRMS is subsonic, referring to the speed of sound of the matter. The US and DS conditions are supersonic, though. This implies that the flow has to cross two sonic points across the shock, the first going from supersonic to subsonic flow and the second going vice versa. The first point is hydrodynamically unstable, and a subshock occurs at this point, which is mediated by other processes (e.g. plasma instabilities) on a length scale much smaller than that of scattering.
- The radiation is highly anisotropic, and exhibits a high energy tail with a typical cutoff energy of \( \sim \Gamma_u^2 m_e c^2 \).

We will elaborate on these points later in this chapter.

### 7.6. Low density shocks

The radiation processes determining the shock structure are Compton scattering, photon-photon pair production and annihilation and Bremsstrahlung emission and absorption. All of these processes are two body interactions except the Bremsstrahlung absorption. All two body mechanisms
scale the same with the density. This scaling is shown explicitly later, in section \( \S \) 8.1, but at this point it is sufficient to notice that scaling the radiation intensity, densities and length scales across the shock by \( n_u \) results in all two body processes being independent on \( n_u \).

As we found in this work, the plasma and radiation state is not far from Compton Equilibrium (CE) across the shock for all upstream energies. It is easy to understand that under these conditions self absorption is unimportant unless the energy density in the radiation is close to the thermal equilibrium energy density \( a_{BB} T^4 \) (\( T \) being the temperature of the plasma and approximately \( 1/3 \) of the average photon energy). Since the radiation energy density is practically limited by its downstream value \( a_{BB} T_d^4 \), the immediate implication is that when \( T > T_d \), even by a factor of a few, self absorption is negligible.

We note that mathematically taking \( n_u \to 0 \) leads to equations that are independent of \( n_u \) around the shock transition since \( T \gg T_d \to 0 \). The solution in such a case gives us a “zero density” profile of the quantities across the shock (e.g. temperature, velocity, densities etc.). This profile has a characteristic minimal temperature \( T_{\text{min}}(\varepsilon) \) in the part of the shock that ends about \( \beta_d^{-1} \) optical depths downstream of the velocity transition. \( T_{\text{min}}(\varepsilon) \) depends only on the upstream energy. In the case where \( n_u \) is such that \( T_d(n_u) \ll T_{\text{min}}(\varepsilon) \), the scaled shock profile around the velocity transient will be the same as the scaled zero density profile, independent of \( n_u \), and we refer to it as a “low density” shock. A general expression for the value of \( n_u \) for which the shock can be considered a low density shock is quite complicated, but as an example, for \( \varepsilon = 10 \text{ MeV} \) we find that \( n_u < 10^{19} \text{cm}^{-3} \) is in the low density limit. The transition value of \( n_u \) increases with increasing \( \varepsilon \), implying that real astrophysical shocks are often in the low density limit, and practically all RRMS shocks are low density shocks. However, we do not limit the numerical solutions only to low density cases.

### 7.7. The conditions under which RMS may exist

For a shock to become radiation mediated, several conditions should be met. Let’s consider a system of length \( L \), proton density \( n \) and a shock with velocity \( \beta c \). The first condition is that the thermal equilibrium DS will be dominated by radiation, Eq. (22). Another condition is that the length of the system must be large enough. We require that the length is sufficient for deceleration,

\[
L \gg L_{\text{dec}} = (\sigma_T n \beta)^{-1}. \tag{39}
\]

The minimum total energy carried by the shock is

\[
E \sim \frac{\beta^2}{2} m_p c^2 n L^3 > \frac{m_p c^2}{2 \sigma_T \beta n^2} \approx 3 \times 10^{29} n_{-20}^{-2} \beta^{-4} \text{erg}. \tag{40}
\]
This immediately implies that for ISM densities, $n < 10^4$, a solar mass rest energy can not drive a RMS. In this case, a shock would be collisionless, i.e. mediated by collective plasma processes with the DS not dominated by radiation. We can also put a lower limit for the mass on the shocked object,

$$M > L^3 n m_p = \frac{m_p}{\sigma_T^2 n^2 \beta^3} \approx 7 \times 10^{50} \left( \frac{n}{\text{cm}^{-3}} \right)^{-2} \left( \frac{\beta}{0.2} \right)^{-3} \text{g.} \quad (41)$$

### 7.8. Physical assumptions

We use, in this work, a single plasma velocity and effective temperature at each point in space, an assumption which should be justified. In fact, for the continuity equation to be correct, the distribution of particles in the rest frame (i.e. the zero momentum frame) needs to be fairly close to isotropic. The isotropization is mediated by processes that act on timescales much shorter than the characteristic scale of changes in the shock, which is the mean time between Compton scatterings. For comparison, we give the plasma time compared to the mean time between Compton scatterings of an electron,

$$\frac{t_{pl}}{t_{scat}} = n_e \sigma_T c = \frac{n_e}{\omega_{pl}} \sqrt{\frac{4 \pi n_e e^2}{m_e^2}} \approx 10^{-9} n_{e,19}^{1/2} n_{\gamma},$$

where $n_e = 10^{19} n_{e,19} \text{ cm}^{-3}$. Given that in the conditions of the immediate downstream the number $n_{\gamma}/n_e$ is not much larger than unity, we see that plasma processes have enough time to be efficient in mediating local equilibrium of the electrons and positrons.

The second assumption we make about the plasma - the existence of an effective temperature, is somewhat more subtle. Theoretically, the electrons and positrons can have a general distribution function which interacts with the radiation within the shock. However, most of the shock is characterized by a strong dominance of radiation energy density over particle thermal energy density. This leads to the electrons and positrons being “held” in momentum space by the radiation, since each scattering changes the energy of the electron considerably, if it departs significantly from the average photon energy. The only way to maintain a very non-thermal electron spectrum is having a radiation spectrum which is not dominated by a typical energy, e.g. a power law. In our results we find that the radiation actually has a typical energy in the rest frame of the plasma, and when a high energy tail appears it has a very low cross section and interaction rate with the plasma. This leads to a conclusion that the plasma can indeed be assumed to have a typical energy, and thus simplify the calculations significantly.

A note is in place here regarding Coulomb collisions. The effective cross section for Coulomb collisions of electrons on protons is $\sigma \sim e^4/\varepsilon_k^2$, where $\varepsilon_k$ is the electron kinetic energy. When the energy of the electron is of the order of $m_e c^2$, the cross section is similar to $\sigma_T$, the Thomson
cross section. This implies that Coulomb collisions in RRMS play a marginal role in equilibrating the motion of particles in the plasma, as the photon density inside the shock is typically of the order of the electron density. Unlike plasma instabilities, this process can not account for the equilibration of the distribution function of the particles. At low energies, i.e. NRMS, Coulomb collisions may become dominant (see Weaver 1976) due to a much larger effective cross section.

8. Formulation of the numerical problem

8.1. Equations

We wish to solve the problem of a steady state planar shock, thus we can construct a set of equations which are time independent in the frame of the shock. The basic equations governing the shock structure are:

1. Conservation of energy flux
   \[ \frac{d}{dz_{sh}} T_{sh}^{0z} = 0, \]  
   (42)

2. Conservation of momentum flux
   \[ \frac{d}{dz_{sh}} T_{sh}^{zz} = 0, \]  
   (43)

3. Conservation of proton number
   \[ n_p = n_{p,u} \frac{\Gamma u}{\Gamma \beta}, \]  
   (44)

4. Production and annihilation of positrons
   \[ \frac{d(\Gamma \beta n_+)}{dz_{sh}} = Q_+ / c, \]  
   (45)

5. And the radiative transfer equation
   \[ \mu_{sh} \frac{dI_{\nu_{sh}}(\mu_{sh})}{dz_{sh}} = \eta_{sh}(\mu_{sh}, \nu_{sh}) - I_{\nu_{sh}}(\mu_{sh}) \chi_{sh}(\mu_{sh}, \nu_{sh}). \]  
   (46)

Here we use the following notations: \( z_{sh} \) is the physical length in the shock frame, \( c\beta \) is the flow velocity in the shock frame and \( \Gamma \) is the corresponding Lorentz factor. \( n_p \) is the rest frame proton density, and \( n_+ \) is the rest frame density of positrons. \( T_{sh}^{\alpha\beta} \) is the stress-energy tensor in the shock frame,

\[ T_{sh}^{\alpha\beta} = T_{sh,pl}^{\alpha\beta} + T_{sh,rad}^{\alpha\beta}, \]  
(47)

where the subscripts \( pl \) and \( rad \) refer to the plasma and radiation, respectively. For the radiation \( T_{sh,rad}^{\alpha\beta} = F_{rad,sh} \) and \( T_{sh,rad}^{zz} = P_{rad,sh} \), as defined in Eqs. (D3) and (D4), respectively. \( Q_+ = \partial n_+ / \partial t \) is the positron production minus annihilation rate in the local rest frame. The shock
frame radiation field $I_{\nu_{sh}}(\mu_{sh})$ is a function of frequency $\nu_{sh}$ and direction $\mu_{sh} = \cos \theta_{sh}$ (where $\theta_{sh}$ is the azimuthal angle with respect to the z axis). The emissivity $\eta$ and absorption $\chi$ are in general functions of the plasma parameters and the local radiation field.

The detailed way in which we calculate $\eta$, $\chi$ and $Q_+$ is specified below. The processes we consider for the solution are Compton scattering, Bremsstrahlung emission and absorption and two photon pair production and annihilation. Other processes, which we neglect in this work, play a less significant role. The most prominent are double Compton ($\gamma + e \rightarrow 2\gamma + e$), three photon pair annihilation ($e^+e^- \rightarrow 3\gamma$) and pair production on nuclei. Other processes such as muon and pion pair production and synchrotron are insignificant to the structure of the shock.

We assume that the plasma is characterized at any point $z_{sh}$ by a temperature $T(z_{sh})$, a velocity $c\beta(z_{sh})$ and densities $n_p(z_{sh})$, $n_+(z_{sh})$. Using that, we can calculate the plasma stress energy tensor and rewrite Eqs. (42) and (43) using these variables.

In this work we use a set of scaled equations. The scaling relations are

$$\hat{T} = \frac{T}{m_ec^2},$$
$$\hat{\nu} = \frac{h\nu}{m_ec^2},$$
$$\hat{x}_+ = \frac{n_+}{n_i},$$
$$\hat{\dot{z}}_{sh} = \Gamma u n_u \sigma T z_{sh},$$
$$\hat{d\tau} = \Gamma (1 + \beta) (n_e + n_+) \sigma T d z_{sh},$$
$$\hat{\dot{I}} = \frac{I \Gamma^2 u \beta u n_u (m_p/m_e) hc}{(48)}$$

where the quantities on the LHS are the scaled values. The equations can now be written in a scaled manner.

First, we rewrite the conservation equations (42) and (43), using Eq. (44) to eliminate the densities. The plasma stress energy tensor in the shock frame is

$$T_{pl,sh}^{0z} = \Gamma^2 \beta (e_{pl} + P_{pl}) ,$$
$$T_{pl,sh}^{zz} = P_{pl} + \Gamma^2 \beta^2 (e_{pl} + P_{pl}) ,$$

where the comoving frame energy density $e_{pl}$ and pressure $P_{pl}$ are given by

$$e_{pl} = n_i m_pc^2 + (n_e + n_+) m_e c^2 + \frac{3}{2} f(T) (n_e + n_+) T ,$$
$$P_{pl} = (n_e + n_+) T ,$$

where we neglected the thermal pressure of the protons. Here we introduced the equation of state parameter for the electrons and positrons, $f(T)$. It is clear that for NR plasma $f = 1$ and
for relativistic plasma \( f = 2 \). We interpolate between these values using a fit to an exact solution for the equation of state for a Maxwellian distribution of velocities

\[
f(T) = \frac{1}{2} \tanh \left( \frac{\ln(T/m_e c^2) + 0.3}{1.93} \right) + \frac{3}{2}
\]

This fit is good to \( \sim 2 \times 10^{-3} \) everywhere. The exact calculated value of \( f \) as well as the fitted function are shown in fig. 1.

The two equations of conservation can now be written explicitly

\[
\Gamma \Gamma_u \left\{ 1 + (1 + 2 x_+ \frac{m_e}{m_p} \left[ 1 + \hat{T} \left( 1 + \frac{3}{2} f(\hat{T}) \right) \right] \right\} + 2 \pi \hat{F}_{rad,sh} = 1 + \frac{m_e}{m_p},
\]

\[
\Gamma \beta \Gamma_u \hat{u} \left( 1 + (1 + 2 x_+ \frac{m_e}{m_p} \left[ 1 + \hat{T} \left( \frac{1}{(\Gamma \beta) \hat{T}^2} + 1 + \frac{3}{2} f(\hat{T}) \right) \right] \right) + \frac{1}{\beta u} 2 \pi \hat{P}_{rad,sh} = 1 + \frac{m_e}{m_p},
\]

where

\[
\hat{F}_{rad,sh} = \frac{F_{rad,sh}}{2 \pi \Gamma_u^2 \beta_n n_u m_p c^3},
\]

\[
\hat{P}_{rad,sh} = \frac{c P_{rad,sh}}{2 \pi \Gamma_u^2 \beta_n n_u m_p c^3},
\]

are the scaled energy and momentum fluxes of the radiation field.

The transfer equation [Eq. (46)] can be written as

\[
\mu_{sh} \frac{d \hat{I}_{sh} (\mu_{sh})}{d \tau_{sh}} = \hat{\eta}_{sh} (\mu_{sh}, \nu_{sh}) - \hat{I}_{sh} (\mu_{sh}) \hat{\chi}_{sh} (\mu_{sh}, \nu_{sh}),
\]

where, in general, the emissivity and absorption coefficients are the sum of the contributions due to the processes considered

\[
\hat{\eta}_{tot} (\mu, \nu) = \sum \hat{\eta}_{proc} (\mu, \nu),
\]

\[
\hat{\chi}_{tot} (\mu, \nu) = \sum \hat{\chi}_{proc} (\mu, \nu).
\]

The transformation relations for the scaled emissivity and absorption are

\[
\hat{\eta} = \frac{\eta}{\Gamma (1 + \beta) \sigma_T (n_e + n_+)} \frac{m_e}{m_p \Gamma_u^2 \beta_n n_u h c},
\]

\[
\hat{\chi} = \frac{\chi}{\Gamma (1 + \beta) \sigma_T (n_e + n_+)}.\]

The equation for the number of pairs can be rewritten as

\[
\frac{dx_+}{d \tau_{sh}} = \hat{Q}_+,
\]

where the scaled rate of pair production is

\[
\hat{Q}_+ = \frac{Q_+}{\Gamma^2 \beta (1 + \beta) m_p (n_e + n_+) \sigma_T c}.
\]

We describe next the contribution of the various radiation processes to the positron number equation Eq. (45) and the transfer equation Eq. (46).
Fig. 1.— The equation of state parameter $f$ as a function of $\hat{T}$: A numerical calculation for a Maxwellian distribution is shown in solid, and the approximated expression (53) is shown by dashed line.
8.1.1. Compton scattering

In general, the Compton contribution is given by

\[ \eta_s(\mu, \nu) = (n_e + n_+ + n_-) \int d\Omega' d\nu' \frac{d\sigma_s}{d\nu' d\Omega'} (\nu', \Omega' \rightarrow \nu, \Omega) I_{\nu'}(\Omega'), \]

(65)

\[ \chi_s(\mu, \nu) = (n_e + n_+ + n_-) \int d\Omega' d\nu' \frac{d\sigma_s}{d\nu' d\Omega'} (\nu, \Omega \rightarrow \nu', \Omega') \]

(66)

where the total cross section is

\[ \sigma_c(\nu, T) = (n_e + n_+ + n_-) \int d\Omega' d\nu' \frac{d\sigma_s}{d\nu' d\Omega'} (\nu, \Omega \rightarrow \nu', \Omega'). \]

(67)

The normalized emissivity and absorption are

\[ \chi_s(\mu, \nu) = \frac{1}{\Gamma(1 + \beta)} \int d\Omega' d\nu' \frac{d\sigma_s}{d\nu' d\Omega'} (\nu', \Omega' \rightarrow \nu, \Omega) I_{\nu'}(\Omega'), \]

(68)

\[ \left[ \Gamma(1 - \beta \mu_{sh}) \right]^{-1} \hat{\chi}_{s,sh}(\nu_{sh}) = \hat{\chi}_s(\nu) = \frac{1}{\Gamma(1 + \beta)} \hat{\sigma}_s(\nu, \hat{T}), \]

(69)

where \( \hat{\sigma} \equiv \sigma/\sigma_T \) is the Thomson normalized cross section and the transformations between the shock frame and rest frame values of \( \nu, \mu \) are given in the appendix.

8.1.2. Pair production and annihilation

Pair annihilation The photon emission arising from annihilation of pairs has the form

\[ \eta_\nu = \frac{1}{4\pi} \eta_{\nu h\nu} = \frac{\hbar \nu m_n n_+ \sigma_T f_+(\nu, T) r_+(T)}{4\pi}, \]

(70)

where \( r_\pm \) is a dimensionless function of \( T \) accounting for the rate of annihilation and \( f_\pm \) is the spectral distribution of the photons, where

\[ \int f_\pm(\nu, T)d\nu = 1. \]

(71)

The implementation we use for \( f_\pm \) is based on Zdziarski (1980), which fits a function to the results of Monte Carlo simulations for the emitted spectrum. For the annihilation rate we use, based on Svensson (1982),

\[ r_\pm(\hat{T}) = \frac{3}{4} \left[ 1 + \frac{2\hat{T}^2}{\ln(2\eta_{\nu E} \hat{T} + 1.3)} \right]^{-1}, \]

(72)

where \( \eta_{\nu E} = e^{-\gamma E} \approx 0.5616, \gamma_E \approx 0.5772 \) is Euler’s constant. The normalized emissivity is, based on Eq. (70), given by

\[ \left[ \Gamma(1 - \beta \mu_{sh}) \right] \hat{\eta}_{\pm,sh}(\nu_{sh}, \Omega_{sh}) = \hat{\eta}_{\pm}(\nu, \Omega) = \frac{(x_+ + 1)x_+ \nu f_\pm(\nu, \hat{T}) r_\pm(\hat{T}) m_n}{4\pi(2x_+ + 1)\Gamma_0 \Gamma^2 \beta(1 + \beta) m_p}. \]

(73)
The annihilations rate in Eq. (45) is simply
\[ \dot{Q} = -\frac{1}{2} n_e n_n \sigma_T c r_\pm (T), \]  
and the scaled contribution to Eq. (63) is
\[ \dot{Q}_+ = -\frac{x_+ (x_+ + 1) r_\pm (T)}{2\Gamma^2 \beta (1 + \beta) (1 + 2x_+)} . \]  

**Pair production**

The two photon pair production contribution to the absorption in the transfer equation is
\[ \chi_{\nu, \gamma\gamma} (\mu) = \int \sigma_{\gamma\gamma} (\nu, \nu', \mu, \Omega') \frac{I_\nu (\Omega')}{\nu' \nu} (1 - \cos \theta_1) \Theta [\nu' (1 - \cos \theta_1) - 2v_p^2] d\Omega' d\nu', \]  
where \( \theta_1 \) is the angle between \( \mu \) and \( \Omega' \). The scaled absorption can be written as
\[ \hat{\chi}_{\nu, \gamma\gamma} (\mu) = \frac{\Gamma u m_p}{m_e} \left( \frac{m_e}{m_n} \right)^2 \frac{1}{(1 + \beta) (2x_+ + 1)} \times \int \hat{\sigma}_{\gamma\gamma} (\hat{\nu}, \hat{\nu}', \mu, \Omega') \frac{\hat{I}_{\nu} (\Omega')}{\hat{\nu}' \hat{\nu}} (1 - \cos \theta_1) \Theta [\hat{\nu}' (1 - \cos \theta_1) - 2\hat{v}_p^2] d\Omega' d\nu', \]  
where the frame in which \( \hat{\chi} \) is calculated is simply the frame of \( \hat{I}' \). For the cross section we use [e.g. Padmanabhan (2000)]
\[ \sigma_{\gamma\gamma} (s) = \frac{3 \sigma_T}{8} \left[ \left( 2 + \frac{2}{s} - \frac{1}{s^2} \right) \cosh^{-1} s^{1/2} - \left( 1 + \frac{1}{s} \right) \left( 1 - \frac{1}{s} \right)^{1/2} \right], \]  
where
\[ s = \frac{1}{2} h\nu h\nu' (1 - \mu \mu') \]  
is the center of momentum energy squared. To shorten the computing time we integrate over \( \phi' \) assuming that \( \sigma_{\gamma\gamma} \) changes slowly with \( \phi' \) and that \( \Theta [\nu' (1 - \cos \theta_1) - 2v_p^2] \) has the same value for most \( \phi' \) values, and get approximately
\[ < 1 - \cos(\theta_1) >_\phi = 1 - \mu \mu'. \]

To find the positron production rate \( Q_+ \), we use the rate of photon loss to this process
\[ Q_+ = -\frac{1}{2} \dot{n}_\gamma = \frac{1}{2} \int \frac{I_\nu (\mu)}{\nu} \chi_{\nu, \gamma\gamma} (\mu) d\nu d\Omega. \]  
The scaling of the production rate follows
\[ \hat{Q}_+ = \frac{\Gamma u m_p}{2 m_e} \int \frac{I_{\nu} (\mu)}{\nu} \hat{\chi}_{\nu, \gamma\gamma} (\mu) d\nu d\Omega. \]
8.1.3. Bremsstrahlung

Bremsstrahlung emission includes contributions from $e^- p$ and $e^+ p$ encounters, as well as from $e^- e^-, e^+ e^+$ and $e^- e^+$ encounters, which become important sources of photon production at high temperatures. Assuming the emission is isotropic, we follow Svensson (1982) and obtain

$$
\dot{n}_{\gamma,f f}(\Omega, \nu) = \frac{1}{\pi^2} \sqrt{\frac{2}{\pi}} \alpha_e \sigma_T m_e^{1/2} c^2 n_i^2 \frac{e^{-h\nu/T}}{\sqrt{T\nu}} \lambda_{ff},
$$

(82)

where $\alpha_e$ is the fine structure constant, and

$$
\lambda_{ff}(x_+, T) = (1 + x_+) \lambda_{ep} + \left[ x_+^2 + (1 + x_+)^2 \right] \lambda_{ee} + x_+ (1 + x_+) \lambda_{+^{-}}
$$

(83)

is a numerical factor accounting for the presence of electron-positron pairs and for relativistic correction at high temperature. We use a prescription for bremsstrahlung emission based on Skibo et al. (1995) (note that there is an errata correction to this paper), which gives a general fit for the Gaunt factor as a function of temperature, positron density and the emitted frequency. The transformation between the different notations is

$$
\lambda_{ff} = \frac{\pi}{2\sqrt{3}} g_s,
$$

where $g_s$ is the Gaunt factor as given in Skibo et al. (1995).

The emissivity resulting from Eq. (82) is

$$
\eta_{ff,nu}(\mu) = h\nu \dot{n}_{\gamma,f f}(\Omega, \nu) = \frac{h}{\pi^2} \sqrt{\frac{2}{\pi}} \alpha_e \sigma_T m_e^{1/2} c^2 n_i^2 \frac{e^{-h\nu/T}}{\sqrt{T\nu}} \lambda_{ff}.
$$

(84)

The normalized emissivity then reads

$$
\left[ \Gamma(1 - \beta_{sh}) \right]^2 \dot{\eta}_{ff,sh}(\hat{\nu}_{sh}, \Omega_{sh}) = \dot{\eta}_{ff}(\hat{\nu}, \Omega) = \frac{n_i m_i m_p}{\pi^2 \alpha_e \sigma_T m_e^{1/2} c^2 n_i^2} \frac{e^{-h\nu/T}}{\sqrt{T\nu}} \lambda_{ff}.
$$

(85)

Minimal $\nu$. Bremsstrahlung production has a cutoff at low $\nu$, which is determined Debye screening. Coulomb screening suppresses bremsstrahlung emission at impact parameters larger than the Debye length $\lambda_D = \sqrt{T/4\pi e^2(n_e + n_+)}$, implying a low energy cutoff for bremsstrahlung emission (Weaver, 1976b)

$$
\epsilon_{sc} \simeq \frac{\gamma_{e,th} \beta_{e,th}}{\lambda_D} \hbar c,
$$

(86)

where $\gamma_{e,th}$ is the Lorentz factor associated with the random (“thermal”) motion of the electrons, and $\beta_{e,th}$ is the associated velocity (in units of $c$). Setting $\gamma_{e,th} \simeq 1 + 3T/m_e c^2$ we get for the non relativistic case ($T \ll m_e c^2$)

$$
\epsilon_{sc,nr} \simeq 2.87 \times 10^{-6} n_i^{1/2} (1 + 2x_+)^{1/2} \text{KeV},
$$

(87)

and for the relativistic case ($T \gg m_e c^2$)

$$
\epsilon_{sc,rel} \simeq 9.12 \times 10^{-10} n_i^{1/2} (1 + 2x_+)^{1/2} \left( \frac{T}{\text{KeV}} \right)^{3/2} \text{KeV},
$$

(88)
where $n_i = n_i,15 \times 10^{15} \text{ cm}^{-3}$.

We note that since the code describes in detail upscattering and bremsstrahlung self absorption, there is no need for a special calculation of the lower limit caused by self absorption prior to upscattering, as was done by Weaver (1976), for example. For the same reason there is no need to consider the dynamical time a low energy photon needs in order to be upscattered.

**Bremsstrahlung self absorption.** Using Kirchhoff’s law and the calculated value of $\eta_{\nu,ff}$ in the rest frame of the plasma we simply have

$$\chi_{\nu,ff} = \frac{\eta_{\nu,ff}}{B_{\nu}(T)} [\text{cm}^{-1}],$$

where

$$B_{\nu}(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}$$

is Planck’s spectrum. The underlying assumption is that the distribution function of the electrons and positrons is similar to thermal.

The normalized Plank spectrum is

$$\hat{B}_{\nu} = \frac{2m_e^2c^3}{\hbar^2m_p} \frac{1}{\Gamma_a^2 \beta u n_u} \frac{\hat{\nu}^3}{e^{\hat{\nu}^3/k_B T} - 1},$$

and the normalized absorption is

$$[\Gamma(1 - \beta u)]^{-1} \hat{\chi}_{ff,sh}(\hat{\nu}_sh) = \hat{\chi}_{ff}(\hat{\nu}) = \frac{\hat{\eta}_{\nu,ff}}{B_{\nu}(T)}$$

8.1.4. Summary

To summarize the entire set of equations - the final equations are: (54), (55), (58) and (63), for the set of variables $S = \{ \hat{T}(\tau_s), \beta(\tau_s), x_+(\tau_s), \hat{I}_{\nu,sh}(\mu_{sh})(\tau_s) \}$. The contributions of the radiation processes to the transfer equation [Eq. (58)] are given in Eqs. (68), (69), (73), (77), (85) and (92). The contributions to $\hat{Q}$ in the positron fraction equation [Eq. (63)] are given in Eqs. (75) and (81).

8.2. Boundary conditions

We searched for the solution of the equations given above over a finite range of optical depth around the shock transition that satisfies the following requirements:
• The solution includes a subsonic region downstream of a supersonic region with continuous radiation field $I_{\nu sh}(\mu_{sh})$ and positron flux across the transition point between the two regions.

• The radiation momentum flux in the last several optical mean free paths in the US region is negligible compared to the far US electron momentum flux.

• The width of the subsonic region is sufficiently larger than the optical mean free path, such that the solution is insensitive to the precise boundary conditions that are applied at the DS edge while remaining short enough as to avoid reaching the second supersonic region which exists DS of the subsonic region.

The boundary conditions in the upstream are $I_{\nu sh}(\mu_{sh} > 0, z_{sh} = -\infty) = 0$, i.e. no radiation coming from the upstream\(^1\). In addition, the positron number is taken as 0 at the US boundary.

The boundary conditions at the far downstream are given by thermal equilibrium. Since the calculation does not reach the far DS, we use a boundary condition in the DS which corresponds to isotropy of the radiation field in the rest frame of the plasma, and implement it by a reflector. The radiation that reaches the DS end of the calculation at $\tau_{s} = \tau_{end}, I_{\nu sh}(\mu_{sh} > 0, \tau_{end})$, is reflected back and is used as the boundary condition for the US going photons at that point, $I_{\nu sh}(\mu_{sh} < 0, \tau_{end})$. The reflection is done either in the rest frame of the far DS velocity $c\beta_d$ or in the rest frame of the local plasma velocity that is obtained by solving the conservation equations. These two methods differ only in the close vicinity of the boundary, but the results close to the sonic point are unaffected by the difference. For numerical reasons we multiply the reflected radiation by a factor which is close to unity. Again, the change (within $\sim 10\%$) in this factor has negligible effect on the shock structure near the sonic point.

For the reflector used for the DS boundary condition we impose an upper limit on the photon energy of the reflected radiation, typically $3m_ec^2$. The physical reasoning is that high energy photons that cross this point in the DS either scatter and lose most of their energy (as $\hat{T} \ll 1$ at that point and further away), or more likely produce a $e^+e^-$ pair that is swept with the flow. The high energy photons are thus not expected to be isotropic, hence they are “lost” through the boundary.

9. The numerical method

We briefly describe below the numerical method we use for solving the equations.

\(^1\)In practice we use an effective reflector in the US end of the calculation, to avoid numerical fluctuations and shorten the calculation. It does not affect the shock structure.
9.1. Iteration scheme

The numerical solution of this problem is complicated, and the method we used is a method of iterations. The iteration loop starts with an initial guess for a profile

\[ S^0 = \{ T^0(\tau_*), \beta^0(\tau_*), n^0_+(\tau_*), I_{\nu_h}^0(\mu_{sh}, \tau_*) \} \].

The iteration loop proceeds as follows:

1. We compute \( \eta(S^n) \) and \( \chi(S^n) \) of the radiative transfer equation [Eq. (46)] using the profile \( S^n \) (\( n = 0 \) for the first iteration).

2. We integrate directly Eq. (46) and find \( I_{\nu_h}^{n+1}(\mu_{sh}, \tau_*) \), a new solution for the radiation field.

3. We use Eqs. (42), (43), (44) and (45) with the new radiation field \( I_{\nu_h}^{n+1}(\mu_{sh}, \tau_*) \), to find \( T^{n+1}(\tau_*), \beta^{n+1}(\tau_*), n_+^{n+1}(\tau_*), \) the new hydrodynamic profile, \( S^{n+1} \).

4. We repeat steps 1-3 with \( S^{n+1} \) until we achieve convergence.

The iteration scheme proves to be a relatively stable one for the problems we consider. However, we found that using a partial iteration increases the stability of the scheme when the current profile is far from the actual solution, which is usually the case in the first iterations. The partial iteration is implemented by choosing the next iteration’s profile \( S^{n+1} \) and next iteration’s radiation field \( I_{\nu_h}^{n+1}(\mu_{sh}, \tau_*) \) as a weighted average between the values found for the new iteration and the values in the previous iteration (\( n \)).

Two different methods of solutions were used for the NR and the relativistic cases:

- **NR shocks:** “Flashlight deceleration of cold upstream” - The upstream boundary condition is “cold upstream”: \( I_{\nu_h}(\mu_{sh} > 0, \tau_* = \min(\tau)) = 0, n_+ = 0 \) and \( \beta = \beta_u \). The downstream boundary condition is chosen to be some non zero radiation field \( I_{\nu_h}^{n+1}(\mu_{sh} < 0, \tau_* = \max(\tau)) \) in the upstream direction, which determines the solution. Choosing \( I^I \) which represents the radiation field at some point in the immediate downstream leads to a solution which is the shock profile. We use a Wien spectrum with a temperature lower than \( T_s \) [see Eq. (33)], expected in the DS well behind the velocity transition, and an intensity that satisfies the equilibrium at the DS velocity. Once the resulting solution features an optical depth \( > \beta_d^{-1} \) with DS velocity behind the transition, the choice of \( I^I \) does not affect the results.

- **Relativistic shocks:** “upstream - downstream iterations” - As was mentioned previously, we have found that in the relativistic case the shock structure includes a subshock. We use the subshock, placed at \( \tau_* = 0 \), to divide the shock into two parts: upstream and downstream, that meet at the subshock. The upstream is iterated with the “flashlight” scheme, with \( I_{\nu_h}^{n+1}(\mu_{sh} < 0, \tau_* = 0) \) given by the radiation emitted from the downstream
in the upstream direction. The downstream boundary condition \( I_{\nu,sh}(\mu_{sh} > 0, \tau_s = 0) \) is taken as the radiation emerging from the upstream solution in the downstream direction. The other end has a boundary condition of a reflector, where the flow is approximately in constant velocity and the radiation is almost isotropic in the rest frame. The iterations are carried until the upstream and downstream are consistent with each other, in addition to being self consistent.

9.2. Discretization

We use a discretization of the radiation field \( I_{sh,\nu,sh}(\mu_{sh}) \) as follows

\[
I_{sh,\nu,sh}(\mu_{sh}) = \sum \hat{I}_{sh,ij} \cap (\nu_{sh}, \nu_{sh,i}, \nu_{sh,i+1}) \cap (\mu_{sh}, \mu_{sh,j}, \mu_{sh,j+1}),
\]

(93)

where \( \cap(x, x_1, x_2) = \Theta(x - x_1)\Theta(x_2 - x) \) is a top-hat function. The distribution of \( \nu_{sh,i} \) is logarithmic in the range \( \nu_{\text{min}} \) to \( \nu_{\text{max}} \). Typical values are \( h\nu_{\text{max}} = 10^2 m_e c^2 \) and \( h\nu_{\text{min}} = 10^{-8} m_e c^2 \). The distribution of \( \mu_{sh,j} \) is set to account for relativistic beaming of the radiation in the shock frame as well as a relatively isotropic component in all frames, from US to DS. This is done by a logarithmic separation of \( \mu_{sh} \) in the US direction between \( \mu_{sh} = 0 \) and \( \mu_{sh} = 1 \), with \( 1 - \max(\mu_{sh}) < \Gamma_u^{-2} \). The \( \mu_{sh} < 0 \) directions are chosen as the zeros of a Legendre polynomial, the same as the common Gaussian quadrature. A typical division is shown in fig. 2. We note that in order to account correctly for the relativistic beaming using Gaussian quadrature, for instance, would require a much larger number of azimuthal directions for high values of \( \Gamma_u \).

9.3. Test problems

The numerical scheme and its implementation were tested thoroughly to ensure the results are valid. We present only two of the tests, demonstrating the suitability of the numerical scheme for dealing with repeated Compton scattering and pair production and annihilation. The test results are compared with analytic solutions and are shown to reproduce them well.

9.3.1. Comptonization in a cloud of low and medium optical depth

A thin, stationary planar layer of plasma with Thomson optical depth \( \tau_T \) in the \( z \) direction (perpendicular to the symmetry plane) and a given temperature \( T \) is irradiated at one end, \( \tau_s = 0 \), by

\[
I_\nu(\mu > 0, \tau = 0) = I_0 \delta(\nu - \nu_0) \delta(\mu - 1),
\]

(94)
i.e. a $\delta$ function in $\nu$, directed along the $z$ axis. At $\tau_*=\tau_T$ the boundary condition is $I(\mu<0,\tau_*=\tau_T)=0$. For $\tau_T \leq 1, \hat{T}>1/4$ and $\hat{\nu}_0 \ll T$, the radiation spectrum that is expected to exist inside the cloud is a power law between $\hat{\nu}_0$ and $\hat{\nu}_{\text{max}} \sim \hat{T}$:

$$I \propto \nu^{\log(\tau)/\log(16\hat{T}^2)},$$  \hspace{1cm} (95)

where the single scattering energy boost for a low energy photon is $\sim 16\hat{T}^2$. An exponential decay is expected above $\hat{\nu} \sim 3\hat{T}$. The power law results from multiple Compton scatterings combined with photons escaping through the boundaries. In order to reach the solution for the radiation field, the iteration scheme of the radiative transfer equation is used until the radiation field converges.

The results of two tests with $\hat{T}=1, \hat{\nu}_0=10^{-4}$ are presented, one with $\tau_T=1$ and the second with $\tau_T=0.01$. The resulting spectra inside the cloud are shown in Figs. 3 and 4 for $\tau_T=1$, $\tau_T=0.01$ respectively. In each figure a line representing the theoretically expected power law is shown for reference.

9.3.2. Pair quasi equilibrium for given $T$

This test checks the numerical (integral) pair production and annihilation. We use a setup with a given Wien spectrum of the radiation field

$$I_{\nu}(\mu) \propto \nu^2 e^{-\nu/\hat{T}}.$$ \hspace{1cm} (96)

For a given $\hat{T}$, we find the equilibrium value of $x_+=n_+/n_p$ for which the positron production and annihilation rates cancel each other analytically and numerically. A comparison between the two values obtained is given in table 1 for different temperatures.

Note that $x_+$ does not necessarily grow with $\hat{T}$, since we use different densities $n$ for convenience. We obtain an accuracy of a few % expect for very low temperature, where higher resolution is needed in order to account for the exponential cutoff near $\hat{\nu}=1$. The resolution used here is $v_{n+1}/v_n = 1.4, N_\mu = 12$.

10. Numerical results

In this section we present the numerical results, solving equations (42)-(46) self consistently for different values of the upstream Lorentz factor $\Gamma_u$. We divide the presentation of the results into 2 parts: The structure (§ 10.1) and the radiation spectrum (§ 10.2). The structure is the spatial distribution of integral parameters such as temperature, velocity (or Lorentz factor), pair density and radiation pressure. The spectrum is the distribution of radiation intensity at different angles
Fig. 2.— $\Gamma_u = 20$, distribution of 18 $\mu'$s in three frames: shock frame, US frame and $\Gamma = 10$ frame.

<table>
<thead>
<tr>
<th>$\tilde{T}$</th>
<th>$x_{\text{analytic}}$</th>
<th>$x_{\text{num}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>550</td>
<td>425</td>
</tr>
<tr>
<td>0.5</td>
<td>541</td>
<td>500</td>
</tr>
<tr>
<td>0.8</td>
<td>421</td>
<td>421</td>
</tr>
<tr>
<td>1.5</td>
<td>259</td>
<td>266</td>
</tr>
<tr>
<td>10</td>
<td>43</td>
<td>43</td>
</tr>
</tbody>
</table>

Table 1: Equilibrium values of $x_+$, balancing the pair production and annihilation rates at different temperatures.
Fig. 3.— The radiation spectrum inside a cloud of plasma with $\tilde{T} = 1$ and width $\tau_T = 1$. The radiation entering the cloud has a single frequency $\tilde{\nu}_0 = 10^{-4}$. The line marked PL is the theoretically expected power law, $\tilde{\nu}^0$.

Fig. 4.— The radiation spectrum inside a cloud of plasma with $\tilde{T} = 1$ and width $\tau_T = 0.01$. The radiation entering the cloud has a single frequency $\tilde{\nu}_0 = 10^{-4}$. The line marked PL is the theoretically expected power law, $\tilde{\nu}^{-1.66}$. 
and photon energies (at given locations across the shock), measured in a specific reference frame. Two important frames of reference are the shock frame, in which the solution is steady state, and the local rest frame of the plasma, which is useful for understanding the interaction between the radiation and the plasma.

10.1. Structure

Results are given for $n_u = 10^{15}$ cm$^{-3}$, over regimes where bremsstrahlung absorption is negligible (i.e. they are in the low density limit). The shock profiles can be divided to 4 regions:

1. Far upstream - The velocity is constant, while the radiation intensity and positron fraction grow exponentially until they hold a significant fraction of the energy and momentum of the flow.

2. The transition - Here the flow decelerates considerably, reaching a velocity close to the downstream velocity. For RRMS this regime is bound by a subshock.

3. Immediate downstream - The flow approximately stays at constant velocity, while the plasma and radiation are in CE. A gradual cooling by bremsstrahlung emission and inverse Compton takes place. This region produces the radiation that diffuses upstream and decelerates the incoming plasma.

4. Far downstream - Approximately $\beta_d^{-1}$ optical depths into the downstream, from where most photons can not diffuse upstream. From this point on, a slow thermalization takes place accompanied by a slow decline in the plasma temperature and photon energies, ending when the temperature reaches the downstream temperature. Here bremsstrahlung absorption takes over and thermal radiation at equilibrium is established.

We do not solve the equations in the fourth region since the solution there is straightforward (the radiation is isotropic and in equilibrium with the plasma). Also, note that since the far downstream is supersonic, a second sonic point is expected in RRMS. This, however, is a stable point with no special physical significance.

Figures 5 and 6 show, for different values of $\Gamma_u$, the structure of the relativistic velocity $\Gamma \beta$ across the shock. It can be seen that the deceleration length in units of $\tau_*$ grows with $\Gamma_u$ in a manner faster than linear. The sonic point is robustly found to be a subshock, causing an immediate deceleration of $\delta(\Gamma \beta) \sim 0.1$. Behind the subshock, the velocity approaches its far DS value in a few Thomson optical depths. The last optical depth is affected by the boundary conditions imposed on the right hand side.
Figures 7 and 8 show, for different values of $\Gamma_u$, the structure of the positron to proton number ration, $x_+$, across the shock. The growth of $x_+$ as a function of $\tau_*$ when approaching the subshock is super exponential, and its value reaches a maximum a few optical depths behind the subshock. The maximal value is approximately linear in $\Gamma_u$. Figure 9 shows $x_+\hat{T}$ across the shock, which represents the pressure of the positrons and their relative importance in setting the speed of sound in the plasma, compared to the protons. The value of $x_+\hat{T}$ goes above a few hundreds at the subshock for $\Gamma_u \geq 6$.

Figures 11 and 12 show, for different values of $\Gamma_u$, the structure of the temperature $\hat{T}$ across the shock. The far US shows an exponential growth of $\hat{T}$ as a function of $\tau_*$. The temperature then saturates at a maximum which is approximately linear in $\Gamma_u$, and then decreases towards the subshock. Behind the subshock the temperature jumps, reaching a value of $\hat{T}_{\text{jump}} \sim 0.5$, which grows with $\Gamma_u$, and then cools with a typical distance of a few Thomson optical depths ($\tau_*$).

Figures 13 and 14 show the relativistic velocity $\Gamma\beta$, the temperature $\hat{T}$ and $x_+$ as a function of the scaled distance $\hat{z}_{sh}$, for $\Gamma_u = 10$. These figures illustrate that the shock width is comparable to the upstream Thomson mean free path, as $\hat{z}_{sh}$ is measured in these units.

Figure 10 shows the ratio of thermal energy flux carried by electrons and positrons to the radiation energy flux, $F_{sh}$, vs. $\Gamma\beta/(\Gamma_u/\beta_u)$. The energy flux (“taken” from the protons) is dominated by thermal and rest mass energy flux of the electrons and positrons during most of the transition rather than by radiation energy flux. The energy is given to the radiation when the flow approaches the DS velocity, and the two fluxes are comparable around the subshock. Comparing the results at a fixed point (e.g. $\Gamma = \Gamma_u/2$), this ratio grows with $\Gamma_u$. 
Fig. 5.— The relativistic velocity of the flow $\Gamma \beta$ vs. $\tau_*/\Gamma_u$ for different values of $\Gamma_u$, from the US to the subshock.

Fig. 6.— The relativistic velocity of the flow $\Gamma \beta$ vs. $\tau_*$ for different values of $\Gamma_u$, around the subshock. Notice that the last optical depth on the right hand side is influenced by the boundary conditions, however the flow near the subshock is not affected by this boundary condition.

Fig. 7.— The positron to proton ratio $x_+$ vs. $\tau_*/\Gamma_u$ for different values of $\Gamma_u$, from the US to the subshock.

Fig. 8.— The positron to proton ratio $x_+$ vs. $\tau_*$ for different values of $\Gamma_u$, around the subshock.
Fig. 9.— The rest frame normalized positron pressure $x_+ \hat{T}$ vs. $\tau_*/\Gamma_u$ for different values of $\Gamma_u$, from the US to the subshock.

Fig. 10.— The ratio of thermal energy flux carried by electrons and positrons to the radiation energy flux $F_{\text{pair}}/F_{\text{rad}}$ vs. $\Gamma \beta / (\Gamma_u \beta_u)$ for different values of $\Gamma_u$.

Fig. 11.— The normalized temperature $\hat{T}$ vs. $\tau_*/\Gamma_u$ for different values of $\Gamma_u$, from the US to the subshock.

Fig. 12.— The normalized temperature $\hat{T}$ vs. $\tau_*$ for different values of $\Gamma_u$, around the subshock. Notice that the last optical depth on the right hand side is influenced by the boundary conditions, however the flow near the subshock is not susceptible to it.
Fig. 13.— The relativistic velocity $\Gamma \beta$, the normalized temperature $\hat{T}$ and the positron to proton ratio $x_+$ vs. normalized distance $\hat{z}_{sh}$ for $\Gamma_u = 10$.

Fig. 14.— The relativistic velocity $\Gamma \beta$, the normalized temperature $\hat{T}$ and the positron to proton ratio $x_+$ vs. normalized distance $\hat{z}_{sh}$ for $\Gamma_u = 10$, zoomed on the DS.
10.2. Spectrum

We show the radiation spectrum at different points along the shock profile for the cases $\Gamma_u = 10$ and $\Gamma_u = 30$. The points of interest are:

1. The upstream - where $\Gamma = 0.99\Gamma_u$. At this point we show the spectrum in the rest frame of the plasma (Figs. 15 and 16 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively).

2. The transition - where $\Gamma = \Gamma_u/2$. At this point we show the spectrum in the rest frame of the plasma (Figs. 17 and 18 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively), and in the shock frame (Figs. 19 and 20 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively).

3. The immediate DS - One Thomson optical depth ($\tau_e = 1$) downstream of the subshock. At this point we show the spectrum in the shock frame (Figs. 21 and 22 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively).

![Fig. 15.][1] — The plasma rest frame radiation spectrum $\hat{\nu} \hat{I}_{\hat{\nu}}$ vs. $\hat{\nu}$, for $\Gamma_u = 10$ in the US ($\Gamma = 9.9$).

![Fig. 16.][2] — The plasma rest frame radiation spectrum $\hat{\nu} \hat{I}_{\hat{\nu}}$ vs. $\hat{\nu}$, for $\Gamma_u = 30$ in the US ($\Gamma = 29.7$).

We now give a short description of the main characteristics of the spectrum at different locations across the shock. An extensive analysis and analytic understanding of the results is given in section § 11.

- **Upstream:** The rest frame spectrum is strongly dominated by a photon component beamed in the US direction, with a typical energy of $\sim 3\Gamma_u m_e c^2$, and a much weaker, isotropic component with energy $\sim \Gamma_u m_e c^2$. In the shock frame, the dominant component is beamed in the DS direction and with energy $\sim \Gamma^2_u m_e c^2$. There is also a weaker and not strongly beamed US going component with energy somewhat higher than $m_e c^2$. 

[1]: fig15.png
[2]: fig16.png
• **Transition region:** The radiation in this region is extremely anisotropic in both the shock frame and the rest frame of the plasma. In the rest frame (figs. 17, 18) the radiation is dominated by a high energy component beamed in the US direction, with a typical energy of \( h\nu \approx \Gamma m_e c^2 \), \( \Gamma \) being the Lorentz factor at this point. An isotropic component also exists, which is much weaker in intensity and with typical photon energy similar to the beamed component. In the shock frame (figs. 19, 20) the dominant part of the spectrum is a narrowly beamed component in the DS direction, with typical photon energy \( h\nu \sim \Gamma^2 u m_e c^2 \), and much weaker intensity of US going photons with typical energy of \( h\nu \sim m_e c^2 \).

• **Immediate DS:** In figs. 21, 22 we see that the spectrum can be generally divided to two components: a relatively isotropic component with \( h\nu \sim m_e c^2 \), and a narrowly beamed component going in the DS direction with energy reaching \( h\nu \sim \Gamma^2 u \).

Figure 23 shows the spectrum integrated over \( \mu \) in the immediate DS of \( \Gamma_u = 30 \). It is clear that the energy is dominated by the photons with energies \( \sim m_e c^2 \), but the high energy component is also important. This high energy component is almost completely beamed in a narrow cone with opening angle \( \theta \sim \Gamma^{-1} u \) or somewhat larger, and it holds 10%-20% of the total energy flux of the radiation.

### 10.3. Optical depth due to radiation processes

The dominant mechanisms affecting the radiation in the transition region are Compton scattering and photon-photon pair production. To show the relative importance of the two processes and get a handle on some of the important physical features of the deceleration mechanism, we show the
optical depth for US going and DS going photons in the transition region, for the cases $\Gamma_u = 10$ and $\Gamma_u = 30$. Figures 25 and 26 show the cumulative optical depths for US going photons leaving the subshock and reaching the point where $\Gamma = \Gamma_u/2$ as a function of shock frame frequency. It is clear that many of the photons with $\hat{\nu}_{sh} \gtrsim 1$ will make it from the immediate DS to the middle of the transition, while low energy photons $\hat{\nu}_{sh} \ll 1$ will be scattered on the way.

Figures 27 and 28 show the cumulative optical depths for US going photons with $\hat{\nu}_{sh} \approx 1$ leaving the subshock, vs. the relativistic velocity $\Gamma \beta$ of the flow in the transition region. These photons constitute the majority of the photon flux leaving the immediate DS in the US direction. It can be seen that most of the shock profile, up to $\Gamma \sim 0.9 \Gamma_u$, has a total optical of $\sim 5$ for these photons, most of it due to Compton scattering, and order unity optical depth due to photon-photon pair production.

Figures 29 and 30 show the cumulative optical depths for DS going photons, starting from the point $\Gamma = 3$ in the transition, to the subshock, as a function of $\hat{\nu}_{sh}$. Comparing the results for $\Gamma_u = 30$ and $\Gamma_u = 10$ shows that the optical depth due to both scattering and photon-photon pair production are very similar for both values of $\Gamma_u$, suggesting a common structure and spectrum of upstream going photons in this region.

Figures 31 and 32 show the cumulative optical depths for DS going photons, starting from the point $\Gamma = \Gamma_u/2$ in the transition, to the subshock, as a function of $\hat{\nu}_{sh}$. As was shown earlier, the radiation in the shock frame at the transition region is dominated by photons with energy $\sim \Gamma_u^2 m_e c^2$ directed towards the DS. The figures illustrate that the optical depth for these photons to reach the immediate DS is less than unity. Photons with energies around the pair production threshold in the shock frame $0.1 \lesssim \hat{\nu}_{sh} \lesssim 10$ will suffer a strong attenuation due to
10.4. Numerical convergence of the calculations

10.4.1. Changes in resolution

The solution of the equations is obtained using iterations, as described in § 9.1. Iterations are continued until the integral quantities (e.g. $T$, $\Gamma$, $x_+$, $P_{\text{rad}}$ etc.) show a change between iterations of less than $\sim 1\%$. For each set of discretization in $\nu_{sh}$, $\mu_{sh}$ and $\tau_*$ the solution is slightly different. The resolution we used for the solutions is given in table 2.

It was found that using $\delta \tau_* = 0.2$ changes the results by $\sim 1\%$ compared to $\delta \tau_* = 0.1$, and we conclude that the solution with either is satisfactory. The convergence of the solution with respect to the resolution in $\nu_{sh}$ and $\mu_{sh}$ was tested using solutions with lower and higher resolutions for $\Gamma_u = 10$. The profiles that were obtained are qualitatively the same as those obtained for the parameters given in table 2. We used several properties of the solution to quantify the convergence. The properties we checked were:

### Table 2: The resolution used in the calculation of the shock profile for different values of $\Gamma_u$

<table>
<thead>
<tr>
<th>$\Gamma_u$</th>
<th>$\delta \tau_*$</th>
<th>$\nu_{i+1}/\nu_i$</th>
<th>$N_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.1</td>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>20</td>
<td>0.1</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>30</td>
<td>0.2</td>
<td>2</td>
<td>18</td>
</tr>
</tbody>
</table>
Fig. 23.— The integrated spectrum in the shock frame, $\dot{\nu}_{sh} \int \hat{I}_{sh,\dot{\nu}_{sh}}(\mu_{sh}) d\mu_{sh}$, at different depths inside the immediate DS ($\tau_s > 0$) for $\Gamma_u = 30$.

Fig. 24.— The integrated spectrum in the shock frame, $\dot{\nu}_{sh} \int \hat{I}_{sh,\dot{\nu}_{sh}}(\mu_{sh}) d\mu_{sh}$, in the immediate DS ($\tau_s = 3$) for $\Gamma_u = 10$, $\Gamma_u = 20$ and $\Gamma_u = 30$.

Fig. 25.— Cumulative optical depth of US going photons from the subshock to $\Gamma = \Gamma_u/2$ vs. shock frame frequency $\nu_{sh}$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 10$.

Fig. 26.— Cumulative optical depth of US going photons from the subshock to $\Gamma = \Gamma_u/2$ vs. shock frame frequency $\nu_{sh}$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 30$. 
Fig. 27.— Cumulative optical depth of US going photons with $\hat{\nu}_{sh} = 1$ leaving the subshock vs. $\Gamma\beta/(\Gamma_u\beta_u)$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 10$.

Fig. 28.— Cumulative optical depth of US going photons with $\hat{\nu}_{sh} = 1.1$ leaving the subshock vs. $\Gamma\beta/(\Gamma_u\beta_u)$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 30$.

Fig. 29.— Cumulative optical depth of DS going photons from the point $\Gamma = 3$ to the subshock vs. $\hat{\nu}_{sh}$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 10$.

Fig. 30.— Cumulative optical depth of DS going photons from the point $\Gamma = 3$ to the subshock vs. $\hat{\nu}_{sh}$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 30$. 
Fig. 31.— Cumulative optical depth of DS going photons from the point $\Gamma = \Gamma_u/2$ to the sub-shock vs. $\hat{\nu}_{sh}$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 10$.

Fig. 32.— Cumulative optical depth of DS going photons from the point $\Gamma = \Gamma_u/2$ to the sub-shock vs. $\hat{\nu}_{sh}$, due to Compton scattering and photon-photon pair production, $\Gamma_u = 30$. 
• $T_{\text{jump}}$ - the temperature immediately behind the subshock;
• The maximal $x_+$ value;
• $P_{sh,\text{jump}}$ - the value of the radiation pressure in the shock frame at the subshock;
• $-\tau_*(\Gamma \beta = 5)$, the normalized optical depth upstream of the subshock at which the Lorentz factor drops by $\sim$half;
• $-\tau_{*,nl}$ - the normalized optical depth upstream of the subshock at which the flow crosses from the far US into the transition region, i.e. the nonlinear point (see detailed explanation in § 11.2);

Note, that the value of $-\tau_{*,nl}$ is very sensitive to small changes in resolution since it is set by an exponential decay of very few photons arriving from the immediate DS. However, its significance for the physical properties of the shock is mild, and we use it here merely as a strong indicator of the numerical convergence. The change in the test parameters as a function of the resolution (number of $\nu'$s times number of $\mu'$s) is given in fig. 33. The results were obtained using lower and higher resolutions, separately in $\nu$ and $\mu$ as well as in combinations between the two.

It is clear that the numerical error around the nominal resolution used in this work is $\sim$few percent, except for the most sensitive parameter, $\tau_{*,nl}$, for which the numerical error is around 10%.

10.4.2. Changes in the length of the DS

In order to verify that the boundary conditions imposed on the DS edge of the shock do not have a significant effect on the final results, around the subshock and in the shock transition region, we compare the results shown here to the results with a longer DS behind the subshock. We are limited in extending the DS because of numerical problems, caused by the proximity to a second sonic point. For this reason we only extend the DS of the calculation with $\Gamma_u = 30$, from $\tau_* = 7$ in the standard calculation to $\tau_* = 10$. We get a shock profile which is very close to the standard one, with changes of order of a percent in integral quantities. The only measure which changes by $\sim 3\%$ is $\tau_{*,nl}$. The temperature and velocity profiles of the two calculations are given in figs. 34 and 35.

11. Simplified analytic modeling of RRMS structure

For shock Lorentz factor $\Gamma_u \gg 1$, the structure of the shock shows physical characteristics that can be understood qualitatively.
Fig. 33.— A summary of numerical convergence tests. Results with different resolutions for $\Gamma_u = 10$ are shown vs. the resolution in $N_\nu \times N_\mu$. The y axis shows the values of $T_{jump}$, $x_{+,max}$, $P_{sh,jump}$, $-\tau_*(\Gamma \beta = 5)$ and $-\tau_{*,nl}$ (see text for definitions) divided by their values for the resolution given in table 2.
11.1. The physical picture

The key to understanding the structure of RRMS lies in the immediate DS. We note that the basic arguments leading to this understanding are the same as for NR RMS, as explained in section 7: The plasma must be able to produce the radiation needed for the deceleration within $\sim \beta_d^{-1}$ Thomson optical depths downstream of the velocity transition. The reasoning for that argument is that photons can not diffuse over much larger optical depths against the stream (the velocity in the DS is at most $\beta_d = 1/3$, implying that the diffusion approximation is always valid). In addition, the expected $y$ parameter for Inverse Compton scattering is large, hence the radiation and the plasma at that location should be close to CPE. We show below that the temperatures in this region are bound by $\sim m_e c^2$, mainly since higher temperatures would result in the production of too many $e^+e^-$ pairs.

Once the immediate DS is understood, a simple estimate of the spectrum of photons emanating from this region in the US direction leads to an understanding of the transition region, and eventually to the asymptotic US which has a characteristic linearized set of equations, resulting in an exponential growth of the radiation field when approaching the shock.

Finally, the flow downstream of the immediate DS is smooth and NR. This is the Thermalization tail, which slowly produces the photon density needed for thermal equilibrium, eventually getting to the far DS equilibrium conditions.

We now follow the shock structure from US to DS, and explain quantitatively the resulting structure.
11.2. General structure

• **Far US:** As was shown in 10.2, the radiation, as seen in the rest frame of the US plasma is strongly dominated by a beamed $[\mu_{\text{rest}} \approx -1 + 1/\Gamma_u^2]$, radiation field with photons of typical energy of several times $\Gamma_u m_e c^2$. To understand the main physical properties of this region it is useful to approximate the radiation field as a delta function in energy and direction, going in the US direction. The asymptotic solution for such a radiation field can be easily found to be an exponential growth of the parameters $P_{\text{rad,sh}}$, $F_{\text{rad,sh}}$, $\Gamma_u \beta_u - \Gamma \beta$ and $T$, with the same exponent, $\lambda_{\text{as}}$, and an exponential growth of $x_+$ with an exponent $2\lambda_{\text{as}}$ [see Sagiv (2006)]. An approximate value for $\lambda_{\text{as}}$ is given by

$$\lambda_{\text{as}} \approx 0.28 \frac{1}{\Gamma_u \beta_u n_u \sigma_T} \left( \frac{\sigma_c}{\sigma_T} \right)^{-1},$$

(97)

where $\sigma_c$ is the total cross section for the photons in the rest frame of the plasma.

The relevant exponents calculated for the case $\Gamma_u = 10$, where the rest frame dominant frequency is $h \nu_{\text{rest}} \approx 36 m_e c^2$ are shown in figure 36. The transition from linear to nonlinear evolution occurs at $\tau_{\ast,\text{nl}} \approx -320$.

This exponential growth changes its nature when the radiation and pairs carry a significant ($\gtrsim m_e/m_p$) part of the upstream momentum and energy flux. In the case $\Gamma_u = 20$, the nonlinearity appears at $\tau_{\ast,\text{nl}} \approx -890$. $\tau_{\ast,\text{nl}}$ grows with $\Gamma_u - 1$ (energy per proton) in a manner faster than linear.

• **Transition region:** The transition region is characterized by a large change in the Lorentz factor of the fluid, giving its energy and momentum flux to radiation and thermal energy of $e^+ e^-$ pairs. The temperature of the plasma approximately follows $T(\Gamma) \sim h \nu(\Gamma) \sim \Gamma m_e c^2$, once the radiation dominates the fluxes (at approximately $\Gamma = \Gamma_u/2$).

This is expected, as the radiation drives the temperature into this ”equilibrium” through frequent Compton collisions of the photons dominating the spectrum in the plasma rest frame.

We now estimate the deceleration of the plasma in the region $\Gamma_u \gg \Gamma \gg 1$. In this region, a photon going in the US direction can undergo either Compton scattering or pair production, with comparable probabilities. Also, its first interaction is likely to be its last, since for scattering the resulting photon will be beamed in the DS direction, suffering a $\sim \Gamma^{-2}$ attenuation in interaction rate, while a produced pair drifts with the plasma and rarely annihilates, both due to interaction rates ($\propto \Gamma^{-2}$ due to time and length contractions) and high temperatures that lower the cross section for annihilation.

The source for the original US going photons is the immediate DS, which emits photons that are relatively isotropic in the shock frame and with energy $\sim m_e c^2$. The KN correction for
Compton scattering is $\sigma_c/\sigma_T \sim ((mc^2)^2/2h\nu T) \sim \Gamma^{-2}$. Pair production is mainly due to interactions with DS going photons, their energy (in the shock frame) is typically $\Gamma^2mc^2$, and $h\nu_1h\nu_2 \sim \Gamma^2(mc^2)^2$. The cross section is suppressed, in that case, by a factor $\Gamma^{-2}$.

The number flux of the US going photons as a function of $\tau_*$, where $\tau_* = 0$ is their point of origin will be

$$\tilde{F}_{sh}(-\tau_*) = e^{\int_{\tau_*}^{\infty} a_0 \Gamma(-\tau')^{-2} \, d\tau'} \approx \Gamma(-\tau_*)^{-2}, \quad (98)$$

where $a_0$ is order unity. The last equality arises from the assumption that the deceleration of the plasma originates from the two interactions we consider. The amount of momentum and energy flux that are removed from the plasma per photon is $\propto \Gamma$, so the number flux goes like $\Gamma^{-2}$. Solving Eq. (98) we get

$$\Gamma(-\tau_*) \approx \Gamma_0 (\tau_0 - \tau_*)^{1/2}, \quad (99)$$

where the constants $\Gamma_0 \approx 1$ and $\tau_0 \approx 0$ depend on the details, e.g. the exact spectrum and scattering. Simply taking $\Gamma_0 = 1$ and $\tau_0 = 0$ results in a qualitatively good fit, as can be seen in fig. 37. It is evident that the deceleration, when approaching the subshock, has a universal structure for different $\Gamma_u$ values.

- **Immediate DS:**

We note that given CPE we can find, based on a single variable, all other integral quantities: $\Gamma$, $P_{rad,sh}$, $F_{rad,sh}$, $n_p$, $T$ and $x_+$. To do so we need five equations. We use Eqs. (43), (42), (44). For large chemical potential, $\mu_{ch} \gg T$, pair production equilibrium gives

$$\frac{n_T}{n_\gamma}|_\text{eq} = \int_0^\infty dx x^2 e^{-\sqrt{x^2 + \hat{T}^{-2}}} = \frac{K_2(\hat{T}^{-1})}{\hat{T}^2}, \quad (100)$$

where $K_2$ is the modified Bessel function of the second kind of order 2. Finally we use the isotropy of the radiation in the plasma rest frame to find a relation between $F_{rad,sh}$, $P_{rad,sh}$ and $\Gamma$. We use these equations below to compare the actual state found in the numerical solutions to that obtained assuming CPE.

We compare the immediate downstream found in the calculation with a CPE at the same temperature. The temperature $\hat{T}$ determines all of the parameters ($x_+, P_{rad, g = \Gamma\beta}$). The assumptions are that $\langle \hat{\nu} \rangle = 3\hat{T}$ and the spectrum is close to a Wien spectrum. As can be seen in figure 38, the values immediately behind the jump are far from CPE, and they quickly move towards the CPE values. We bring only the results for $\Gamma_u = 20$, results for different $\Gamma_u$ are qualitatively the same.

It can be seen that there is a systematic deviation from CPE in the solutions. This is expected, as the high energy tail (explored in detail in 11.3) of beamed photons takes its share only in the conservation equations, since its interaction by scattering and pair production is very weakly coupled to the plasma, due to suppression of the cross sections.
The speed of sound in matter gets close to its ultrarelativistic limit of \( c/\sqrt{3} \), larger than the DS velocity of even ultrarelativistic shocks, which is \( c/3 \). The exact expression for the speed of sound is given in Eq. (C8). The speed of sound remains close to its ultrarelativistic value as long as \( 4x_+\hat{T} \gtrsim m_p/m_e \). In fact, the speed of sound exceeds the DS velocity for \( 4x_+\hat{T} \gtrsim 500 \). \( \hat{T} \) and \( x_+ \) reach these values in the immediate DS of mildly and highly relativistic shocks, and since the length scale for a change in integral quantities is of the order of a mean free path, these shocks remain subsonic for at least this range, until the temperature drops to \( \sim 0.2m_e c^2 \), and the positron number drops exponentially.

A final note about the temperature in the immediate DS is the importance of Bremsstrahlung photon production rate at very low energies, several orders of magnitude below \( T \). The estimated \( T_s \) depends on the total production rate, which in turn is dominated by the log-logarithmic divergence of the Gaunt factor for \( h\nu \ll T \) [see e.g. Svensson (1984)]. Not taking into account this effect would lead to an increase in the temperature of the immediate DS by a factor of \( \sim 1.5 \), both in the simplified analysis and in numerical calculations.

- **Far DS:** This region is characterized by an almost constant velocity and a slow growth in photon number that lowers the temperature. It can be divided into two parts: First, the temperature goes down to \( \sim 50 \) keV, and the positrons annihilate with the extra electrons until \( x_+ < 1 \). Then the temperature continues to decrease and the positrons are negligible. First we consider the part in which \( x_+ > 1 \). The low temperature limit of Eq. (100) yields

\[
\frac{n_l}{n_\gamma}(\hat{T} \ll 1) \rightarrow 2\sqrt{\frac{\pi}{8}} \frac{e^{-\frac{1}{\hat{T}}}}{\hat{T}^{3/2}}.
\]  

(101)

Using arguments similar to those used for the immediate DS estimates, we can write an equation for the evolution of photon number

\[
\frac{1}{n_l(\tau_s)} \frac{dn_\gamma(\tau_s)}{d\tau_s} \approx \frac{g_{\text{ff}}(\hat{T})\Lambda_{\text{eff}}(\hat{T})}{\beta_d \sqrt{\hat{T}}},
\]  

(102)

and assuming that most of the energy flux is already in the radiation we can approximate \( n_\gamma\hat{T} = \text{Const.} \) Using this assumption with Eqs. (101) and (102) we get

\[
\frac{d\hat{T}}{d\tau_s} = -2\sqrt{\frac{\pi}{8}} \frac{g_{\text{ff}}(\hat{T})\Lambda_{\text{eff}}(\hat{T})}{\beta_d} e^{-1/\hat{T}}.
\]  

(103)

The flow reaches \( n_{e+} \approx n_p \) when \( \hat{T} \approx 0.06 \), with a weak dependence on \( \Gamma_u \). From Eq. (103) we see that the length scale is set by the lowest temperatures in this range, and reasonable parameters yield \( \sim 10^5 \) optical depths required for the positrons to annihilate.

In the second part a gradual increase in photon number lowers the temperature and reaches thermal equilibrium. The length scale for this part is

\[
L_T = \frac{\beta_d n_{\gamma,\text{eq}}}{Q_{\text{eff,d}}},
\]  

(104)
where \( n_{\gamma, eq} \approx a_B T_d^3 / 2.8 \) is the thermal equilibrium photon density and
\( Q_{\text{eff,d}} \approx g_{\text{eff,d}} \Lambda_{\text{eff,d}} n_{\gamma, r}^2 \sigma_T c / \sqrt{T_d} \) is the photon generation rate in the DS. An estimate for \( L_T \) yields
\[
\frac{L_T}{(\sigma_T n_d)^{-1}} \approx 3 \times 10^6 (3\beta_d) \left( \frac{g_{\text{eff,d}} \Lambda_{\text{eff,d}}}{10} \right)^{-1} \Gamma_u^{-3/4} n_u^{-1/8}.
\]
The temperature drops as a power law
\[
T(\tau) \propto (\tau_0 - \tau)^{-2},
\]
as was shown in Katz et al. (2009).

### 11.3. High energy photon component beamed in the DS direction

As can be seen in Figs. 21 and 22, the immediate DS has a high energy photon component beamed in the DS direction. The origin of the photons in this beam is the transition region, and we use the simplified analysis presented in 11.2 to find the characteristics of the spectrum.

The approximation \( \Gamma(-\tau) = \sqrt{-\tau} \), which fits the numerical results well, leads to the normalized cross section for upstream going photons with energy \( \sim m_e c^2 \) in the shock frame
\[
\frac{-\tau}{-\tau_*} \approx \frac{\sigma_c (n_e + n_+)}{\sigma_T (n_e + n_+)} \approx \alpha_1 \Gamma^{-2} \approx -\frac{\alpha_1}{\tau_*},
\]
where \( \sigma_c, \sigma_{\gamma\gamma} \) are the local cross sections for Compton scattering and pair production, respectively, and \( n_{\gamma, r} \) is the effective number density of DS going photons on which most of the pair production is done. \( \alpha_1 \) turns out to be a number close to 1.

The physical picture is that the photons coming from the immediate DS with energy \( \sim m_e c^2 \) travel in the US direction and suffer extinction due to the two processes we considered. We denote the shock frame intensity of these photons \( I_0(-\tau) \), and its value would be
\[
I_0(-\tau) = I_0(0) e^{-\int_{-\tau}^0 d\tau} = \frac{I_0(0)}{(-\tau_*)^{\alpha_1}} = \frac{I_0(0)}{\Gamma_2^{2\alpha_1}}.
\]
Each scattered photon receives a \( \sim \Gamma^2 \) boost to its energy, and is beamed into a cone with an opening angle \( \Gamma^{-1} \) in the DS direction. Since the losses of the scattered photons on the way to the immediate DS are less than a factor of 2 and depend weakly on the angle and energy of the photon, we find that the spectrum of the high energy beam \( I_B \) can be approximately described as
\[
I_B(\hat{\nu}_{sh}, \theta_{sh}) \propto \hat{\nu}_{sh}^{1-\alpha_1} \Theta(\theta_{sh}^{-1} - \hat{\nu}_{sh}^{1/2}) \Theta(\hat{\nu}_{max} - \hat{\nu}_{sh}),
\]
where \( \hat{\nu}_{max} \approx \Gamma_u^2 \). To verify that this analysis complies with the numerical results we use the computation with \( \Gamma_u = 20 \). Fig. 39 shows the shock frame intensity of a beam with \( \theta_{sh} \approx 10^{-2} \) with different \( \nu_{sh} \) along the shock, normalized to the maximum value at each frequency, vs. \( \Gamma^2 / \nu_{sh} \). We see that the intensity is mostly contributed by the part in the flow in which \( \Gamma^2 \approx 200\hat{\nu}_{sh} \), as the physical picture requires. Fig. 40 shows the shock frame intensity immediately
after the subshock, normalized to its maximum, at different $\tilde{\nu}_{sh}$, as a function of $\theta_{sh} \tilde{\nu}_{sh}^{1/2}$. We see that the structure of the beam is such that the different energies are beamed according to Eq. (108).

11.4. Shocks with $\Gamma_u \to \infty$

We have seen that as $\Gamma_u \to \infty$ some features of the structure approach an asymptotic form. This is true, in particular, for the structure (temperature, positron fraction and velocity) of the deceleration region close to the immediate DS and for the immediate DS. However, we have also seen that the high energy beam becomes more dominant as $\Gamma_u$ grows. The structure of the shock, particularly the immediate DS, may be different if the high energy beam is the dominant carrier of momentum and energy of the radiation. Unfortunately, a full calculation of very high $\Gamma_u$ shocks is beyond our current capabilities, and further investigation is needed.
Fig. 36.— $\Gamma_u = 10$ far US exponential growth of integral quantities. The Cyan lines show $e^{\lambda \tau_*}$ and $e^{2\lambda \tau_*}$, which are the simplified modeling exponentials expected for $P$, $F$ and $T$ ($\lambda$) and $x_+$ ($2\lambda$).

Fig. 37.— The relativistic velocity $\Gamma \beta$ for different values of $\Gamma_u$ and the simplified analytical result for the structure $\Gamma \beta \sim \sqrt{-\tau_*}$ (bold line) vs. $\tau_*$. 
Fig. 38.— Comparison of the immediate downstream with CPE, for the case \( \Gamma_u = 20 \). The dashed curves represent the theoretical values obtained assuming CPE, while the open circles and solid lines represent the values right after the jump and shortly (\( \Delta \tau_s = 1 \)) afterward, respectively.

Note that the state of the plasma and the radiation immediately behind the subshock is not very close to CPE, and it approaches CPE quickly.
Fig. 39.— $I_{sh, \nu_{sh}}$ directed towards the DS vs. $\Gamma^2/\nu_{sh}$, for different high photon frequencies $\nu_{sh}$, $\Gamma_u = 20$.

Fig. 40.— $I_{sh, \nu_{sh}}$ at $\tau_s = 0$, directed towards the DS vs. $\theta_{sh}\nu_{sh}^{1/2}$, for different angles with respect to the $z$ axis in the shock frame $\theta_{sh}$, $\Gamma_u = 20$. 
12. NR RMS revisited

The structure of NR RMS was solved by Weaver (1976), using the diffusion approximation. The primary goal is to check our scheme and reproduce his results. As our scheme includes a detailed description of the radiation field, we are also able to check the quality of Weaver’s approximations: Radiation field close to isotropy in the rest frame and a spectrum dominated by photons in a narrow energy range.

12.1. Numerical results

The calculation we have performed contains a fairly accurate description of the radiation field and its interaction with matter. The velocity and temperature were calculated under the approximation that the plasma temperature is somewhat lower than CE with local radiation. This approximation is justified in the case where \( n_\gamma / n_e \gg 1 \), where \( n_\gamma \) is the number density of photons, which holds in the transition region (when the energy density of the radiation is a fair fraction of the flow) and the downstream of a NR RMS. We bring here the solution for a shock with upstream energy per proton \( \varepsilon = 50 \text{ MeV} \) and a very low density \( n_u = 10^6 \text{ cm}^{-3} \), which ensures that bremsstrahlung absorption remains unimportant until after the velocity has already reached its downstream value. In the calculation shown here absorption is everywhere unimportant, since it does not reach the downstream temperature.

![Graph](image)

Fig. 41.— The shock structure for \( \varepsilon = 50 \text{ MeV} \). The dotted black line is the analytic solution for \( \Gamma / \beta \) obtained by Weaver 1976 [equation (5.10) there], with average Compton cross section \( \bar{\sigma}_C = 0.56 \sigma_T \).
Fig. 42.— Spectra of the radiation in the shock frame along the shock profile for \( \varepsilon = 50 \text{MeV} \).
Upper left: far upstream \((\beta = 0.99 \beta_u)\), upper right: inside the velocity transition \((\beta = 0.5 \beta_u)\) and lower: in the immediate downstream \((\tau_* = 37)\).

12.2. Comparison with previous work

Figure 41 shows the structure \((\Gamma \beta, \hat{T} \text{ and } \hat{P})\) of a shock with \( \varepsilon = 50 \text{MeV} \) as a function of \( \tau_* \). The dotted black line is the analytic solution for \( \Gamma \beta \) obtained by Weaver 1976 [equation (5.10) there].

We use the average Compton cross section \( \bar{\sigma}_C = 0.56 \sigma_T \), suitable for \( h\nu \approx 0.5 m_e c^2 \), which is the typical photon energy in the transition region. We see that the two lines (Weaver’s and the numerical solution) differ near the immediate DS. This is due to a lower average photon frequency there, compared to the transition region, which increases \( \bar{\sigma}_C \) in the numerical calculation. The maximal temperature reached in the numerical calculation is \( \sim 60 \text{keV} \), in accordance with the results of Weaver 1976 [e.g. figure (6) there].

Looking at the radiation spectra found in our numerical calculations, fig. 42, we are able to
verify two of Weaver’s assumptions. First, it is clear that the spectrum at each point along the shock is dominated by a single photon energy in a narrow energy range. Second, the anisotropy of the radiation is of order $\beta$, which is the expected anisotropy due to diffusion of the radiation.

To conclude, we have shown here that the solution found using our numerical scheme is consistent with Weaver’s results. In addition, the detailed spectra we found support Weaver’s approximations regarding the radiation spectrum inside the shock. It is worthwhile to mention that our numerical scheme was designed and optimized for the solution of the relativistic problem, and is not efficient and easy to use on NR problems. The fact that the results for NR shocks are in agreement with previous work is an indicator that the numerical scheme is correct.

13. Application to Supernova Shock Breakouts

Once an expanding radiation mediated shock reaches a point where the residual optical depth is of order $\tau \sim \beta^{-1}$, whether at the outer shell of a star’s envelope or in the surrounding wind (in case of an optically thick wind), the radiation escapes, and the shock no longer sustains itself. In this section we summarize the relations between the stellar/wind parameters and the velocity, energy and duration of the resulting shock-breakout radiation outburst.

13.1. A general relation between the breakout energy, velocity, and radius

Consider a shock with velocity $\beta_u \gtrsim 0.1$ approaching the photosphere of a spherical mass distribution. The break-out will occur once the optical depth down to the photosphere is $\tau \sim \beta^{-1} \lesssim 10$.

The energy carried by the photons in a shell of shocked material with a width of the order of the shock width is roughly given by

$$\delta E_\gamma = f_E \delta m \beta_u^2 c^2 / 2, \quad (109)$$

where $\delta m$ is the mass of the shell and $f_E$ is a coefficient that depends on the velocity and on the geometry. For a shocked shell in the downstream of a non-relativistic planar shock, $f_E = 36/49$. $f_E$ is of order unity for non relativistic and mildly relativistic shocks in spherical geometry, as long as the distance to the photosphere is not much larger than the radius. The mass of a shell that has an optical depth $\tau$ is roughly given by

$$\delta m \sim 4\pi R^2 \tau \kappa \approx 10^{26} R_{12}^2 (\kappa / \kappa T)^{-1} \tau_{0.5} g, \quad (110)$$

where $\kappa T = \sigma T / m_p \sim 0.4 \text{ cm}^2 \text{g}^{-1}$, $\tau = 10^{0.5} \tau_{0.5}$ and $R = 10^{12} R_{12} \text{ cm}$. Assuming that the energy of shocked plasma with optical depth $\tau \sim 3 \tau_{0.5}$ is emitted, we find that there is a simple relation between the released energy, breakout radius and shock velocity at breakout,

$$\delta E \sim 0.5 \delta m (\beta_u c)^2 \sim 4 \times 10^{46} \beta_u^2 R_{12}^2 (\kappa / \kappa T)^{-1} \tau_{0.5} \text{ erg}. \quad (111)$$
### 13.2. Breakout Dynamics

We use results for the breakout of a shock through the outer layer of a star with a power law density distribution as a function of the distance from the edge of the star (e.g. Matzner & McKee 1999; Katz et al. 2009). Characterizing the density profile with

\[ \rho = \rho_1 \delta^n, \]  

where \( R_* \) is the stellar radius and

\[ \delta = (R_* - r)/r. \]  

For an efficiently convective envelope we have \( n = 3/2 \), suitable for Red Super Giants (RSGs), and for a radiative envelope \( n = 3 \), suitable for BSGs and WR stars. The velocity amplification factor, defined as the ratio between the velocity of the shock at breakout and the average ejecta velocity of the self similar shell \( v_{ej,b} \approx \sqrt{E_{ej}/M_{ej}} \), was found in Katz et al. (2009). The velocity amplification factor for \( n = 3 \) is

\[ A_{v_s} \approx 11(M_{Jup}/M_\odot)^{0.14}(\kappa/\kappa_T)^{0.14}R_{12}^{-0.28}R_{0.5}^{-0.14}, \]  

and for \( n = 3/2 \) it is

\[ A_{v_s} \approx 4(M_{Jup}/M_\odot)^{0.11}(\kappa/\kappa_T)^{0.11}R_{13}^{-0.23}R_{0.5}^{-0.11}. \]  

Assuming typical bulk envelope velocities of \( v_{ej,b} \approx \sqrt{E_{ej}/M_{ej}} \sim 3 - 10 \times 10^8 \) cm/s, or \( \beta_{ej,b} \sim 0.01 - 0.03 \), the typical shock velocities at the last few optical depths of BSGs and WR stars may reach values of \( \beta_u \gtrsim 0.2 \), for which temperatures exceeding \( T \sim 10 \) keV may be reached in the shock transition region [see eq. (32)]. The shock velocities in the wind may be up to twice higher than the maximal shock velocity achieved within the star.

### 13.3. Breakout X-ray Characteristics

Let us consider the radiation that may be emitted during the break-out of a shock from the stellar surface. We consider the emission during a time interval \( R_* / c \) following the time at which the shock reached the surface (as long as the star is not resolved, any emission will be spread in time over this time scale). On similar time scales, the outer part of the star expands significantly, over a distance \( \beta_f R_* \sim R_* \), where \( \beta_f \approx 2 \beta_u \) is the final velocity reached by a fluid element shocked at \( \beta_u \) (following its acceleration due to the post-shock adiabatic expansion, e.g. Matzner & McKee 1999). On the same time scale, \( R_* / c \), photons that originated at an optical depth \( \tau_{esc} \sim \delta^{-1/2}(\tau = 1) \) are capable of escaping. A more detailed calculation gives:

\[ \tau_{esc} \sim 10(M_{Jup}/M_\odot)^{1/8}(\kappa/\kappa_T)^{1/8}R_{12}^{-1/4}. \]
for BSG typical parameters, and

$$\tau_{\text{esc}} \sim 15(M_{\text{ej}}/M_{\odot})^{1/5}(\kappa/\kappa_T)^{1/5} R_{13}^{-2/5}$$ (117)

for RSG parameters. For $1 \gtrsim \beta_u \gtrsim 0.1$, these values are not much larger than the shock widths, $\beta_u^{-1}$, and a non negligible fraction, $\beta_u^{-1}/\tau_{\text{esc}}$, of the photons in these regions are likely to escape over $\sim R_*/c$.

Note that most of the post-shock material has enough time to produce the photons required by thermal equilibrium. As the radiation observed at a given time is the sum radiation emitted from different positions of the star at different stages of the breakout, we expect to see a spectrum which is a sum of Wien spectra with temperatures ranging from the sub-keV equilibrium temperatures $T_{\text{eq}}$ to the tens of keV shock temperatures $T_s$. These considerations also apply, with some changes, to a breakout through an optically thick wind, see Katz et al. (2009).

Finally, the following point should be stressed. By definition, when the shock reaches the breakout radius it propagates within a density profile which varies on a length scale comparable to the shock thickness. An exact calculation of the emerging spectrum requires therefore a time dependent calculation taking into account the modification of the shock structure due to the variation of the density. Our analysis presented above, based on the steady state solutions of the shock structure, is therefore only qualitative.

### 13.4. Implications to recently observed SN X-ray outbursts

The expected breakout in the ultraviolet and X-ray from RMS reaching the surface of a SN progenitor carry valuable information about the progenitor star including a direct measure of the star’s radius. Two recent detections, of SN2006aj and SN2008D (Campana et al. 2006; Soderberg et al. 2008), are particularly interesting because of the relatively high energy of the photon spectrum. Analysis of the later optical SN emission revealed that both were of type Ib/c, probably produced by compact (WR) progenitor stars (Pian et al. 2006; Mazzali et al. 2006; Modjaz et al. 2006; Maeda et al. 2007; Mazzali et al. 2008; Mauresco et al. 2009; Tanaka et al. 2008). While some authors argue that the X-ray outbursts are produced by shock breakouts (Campana et al. 2006; Waxman et al. 2007; Soderberg et al. 2008), others argue that the spectral properties of the X-ray bursts rule out a breakout interpretation, and imply the existence of relativistic energetic jets penetrating through the stellar mantle/envelope (Soderberg et al. 2006; Fan et al. 2006; Ghisellini et al. 2007; Li 2007; Mazzali et al. 2008; Li 2008). (some argue that the breakout interpretation holds for SN2008D, but not for SN2006aj, Chevalier & Fransson 2008)

The main challenge raised for the breakout interpretation was that in both cases, SN2006aj and SN2008D, the observed X-ray flash had a non-thermal spectrum extending to $\gtrsim 10$ keV,
in contrast with the sub-keV thermal temperatures expected (Matzner & McKee 1999). For example, interpreting the observed X-ray spectrum of SN2008D as thermal radiation at $\sim 5$ keV, leads to an estimate of the size of the emitting region, $R \sim 10^{10}$, which is much smaller than the stellar surface (Mazzali et al. 2008). However, the analysis presented here implies that fast, $\beta_s > 0.2$, breakouts may be expected for compact, BSG or WR progenitors, and that for such fast breakouts the X-ray outburst spectrum is expected to be nonthermal, extending to 10’s or 100’s of keV. Let us therefore discuss in some more detail the possible interpretation of the X-ray outbursts associated with SN2006aj and SN2008D as breakouts.

In the case of SN2008D, both the X-ray outburst and early UV/O emission that followed it are consistent with a breakout interpretation (see Soderberg et al. 2008, for detailed discussion). The X-ray burst energy and duration are consistent with the relation of eq. 111, and the UV/O emission is consistent with that expected from the post-shock expansion of the envelope. Moreover, the non-thermal X-ray and radio emission that follow the X-ray outburst are consistent with those expected from a shock driven into a wind by the fast shell that was accelerated by the radiation mediated shock, for wind density and shell velocity, $v/c \sim 1/4$, and energy, $E \sim 10^{47}$ erg, which are inferred from the X-ray outburst. The fast velocity inferred for the late radio and X-ray emission is consistent with that required to account for the non-thermal spectrum.

The case of SN2006aj is more complicated. The X-ray burst energy and temperature are consistent with a mildly relativistic, $v/c \sim 0.8$, shock breakout from a wind surrounding the star, the UV/O emission is broadly consistent with that expected from the expanding envelope, and the non-thermal X-ray emission that follows the X-ray outburst is consistent with that expected from the shock driven into the wind by the fast shell that was accelerated by the radiation mediated shock (Campana et al. 2006; Waxman et al. 2007). However, the duration of the X-ray outburst is larger than expected, the UV/O emission is not as well fit by the model as in the case of 2008D, and the non-thermal radio emission is higher than expected from the wind-shell interaction. Several authors (Campana et al. 2006; Waxman et al. 2007) have suggested that the deviations from the simple breakout model are due to a highly non-spherical explosion. Some support for the non spherical nature of the explosion was later obtained the polarization measurements of Gorosabel et al. (2006) (see, however, Mazzali et al. 2007). Other authors (Soderberg et al. 2006; Ghisellini et al. 2007; Fan et al. 2006; Li et al. 2007) have argued that a relativistic jet is required to account for the observations. We believe that an analysis of the modifications to the simple spherical model, introduced by a highly non-spherical breakout, is required to make progress towards discriminating between the two scenarios.

There is an additional important point that should be clarified in the context of SN2006aj. Under the shock breakout hypothesis, the energy and velocity of the accelerated shell that is responsible for the X-ray emission are $v/c \sim 0.8$ and $E \sim 10^{49.5}$ erg. This is similar to the parameters of the fast expanding shell inferred to be ejected by SN1998bw (associated with GRB080425), based on
long term radio and X-ray emission, which are interpreted as due to interaction with a low density wind (Kulkarni et al. 1998; Waxman & Loeb 1999; Li & Chevalier 1999; Waxman 2004a,b). Long term radio observations strongly disfavor the existence of an energetic, $10^{51}$ erg, relativistic jet associated with SN1998bw (Soderberg et al. 2004). The similarity of SN1998bw & SN2006aj may therefore suggest that such a jet is not present also in the case of SN2006aj. However, the large energy deposited by the explosion in a mildly relativistic shell, $v/c \sim 0.8$, is a challenge in itself: The acceleration of the supernova shock near the edge of the star is typically expected to deposit only $\sim 10^{46}$ ergs in such fast shells (Tan et al. 2001).
Part IV

Discussion

14. CRs from TRSNe

In part § II we discussed the constraints on Galactic sources of high energy CRs between the knee and the ankle, and demonstrated that TRSNe satisfy these constraints.

We derived constraints that must be satisfied by the sources of \( \sim 10^{15} \) to \( \sim 10^{18} \) eV cosmic rays, under the assumption that the sources are Galactic and that the CRs in this energy range are confined by the Galactic magnetic field (as explained in § 5.1, the CRs in this energy range are likely to be confined by the Galactic magnetic field since their Larmor radius is much smaller than the CR disk thickness). In § 5.1 we have shown that the Galactic occurrence rate, \( \dot{N}_s \), of sources producing \( \sim 10^{18} \) eV CRs and their energy production \( E_s \) per event must satisfy the constraints given by Eqs. (7) and (8) \( [E_s \text{ is the energy available for the acceleration of } 10^{18}\text{eV CRs, rather than the total event energy, as explained preceding Eq. (1)}] \). In particular, \( E_s \) must be considerably lower than the kinetic energy of a typical SNe, \( E_s < 0.03\dot{N}_{SN,-2}\zeta^{-1}E_{SN} \), where \( \dot{N}_{SN} = 0.01\dot{N}_{SN,-2}\text{ yr}^{-1} \) is the Galactic SNe rate. \( \zeta \) in these equations is the ratio of the fraction of \( E_s \) deposited in \( 10^{18} \) eV CRs to the fraction of the kinetic SN energy that is deposited in \( \sim 10^9 \) eV CRs. These constraints must be satisfied by any candidate Galactic source. In § 5.2 and § 5.3 we derived additional constraints that are applicable to SNe and TRSNe. These sources are expected to accelerate particles to high energies through the collisionless shocks that they drive into the plasma surrounding the exploding star. In § 5.2 we derived constraints on the velocity of the shells ejected by the explosions [eqs. (13) & (15)], and in § 5.3 we derived constraints on the structure of the ejecta [eqs. (17) & (18)].

In § 6.1 we have shown that ordinary SNe are unlikely to be the sources of \( \sim 10^{18} \) eV CRs, since they are unlikely to be able to accelerate particles to such energy and if they could, they would produce fluxes far exceeding the observed flux at these energies. We have furthermore shown that assuming that the flux of \( \sim 10^{18}\text{eV CRs is produced by the fastest part of the ejecta of SNe, which carry a small fraction of the ejecta energy, would predict a CR flux at lower energy, } \sim 10^{15} \text{ eV, which far exceeds the observed flux.}

In § 6.2 we have shown that Galactic TRSNe may be the sources of high energy, \( \sim 10^{18}\text{eV, CRs. The mildly relativistic shocks driven by such ejecta are likely to be able to accelerate particles to energies exceeding } 10^{18}\text{eV. The estimated rates of TRSNe combined with the typical energy they deposit in mildly relativistic ejecta yield a flux of } 10^{18}\text{eV CRs which is comparable to the observed flux (under the assumption that the efficiency of the acceleration of particles}
in these shocks is similar to that of SNe and that the accelerated particle spectrum is close to \( dN/d\epsilon \propto \epsilon^{-2} \), which, as explained in § 6.2 is supported by radio and X-ray observations. The large fraction, \( \geq 10^{-2} \), of the kinetic energy deposited by such explosions in mildly relativistic ejecta ensures that if TRSNe are responsible for producing the observed flux at \( 10^{18} \text{eV} \) they do not overproduce lower energy CRs. Furthermore, since the TRSNe are a subset of the larger SNe population, their CR production is expected to smoothly join the lower energy CRs produced by SNe.

15. RRMS

In part § III we studied analytically and numerically the problem of RMS, both NR and relativistic.

In section § 7 we discussed the physics of RMS, giving analytic estimates of the length scales and temperatures of the immediate DS of NR RMS [Equations (28), (31) and (33)]. We have shown that the immediate DS of RRMS is expected to be subsonic (in § 7.4), and concluded that the structure of RRMS must contain two sonic point crossings from the US to the DS.

In section § 8 we have formulated the set of equations governing the shock structure and described in detail the approximations used in order to implement them in a numerical scheme, which is described in section § 9.

In section § 10 we have derived numerically self consistent solutions for the profiles and radiation spectra of RRMS, for upstream Lorentz factors \( \Gamma_u \) in the range of 6 to 30. Our main results are:

- **Structure and radiation spectrum:** In § 10.1 we show that the structure of RRMS can be divided to four regions, from US to DS: The far US, the transition region, the immediate DS and the far DS. The far US is characterized by a velocity close to the US velocity and a radiation flux much smaller than the US plasma flux. The transition region is where the velocity (\( \Gamma \beta \)) changes significantly, approaching \( \Gamma \beta \sim 1 \) while the momentum and energy flux is transferred to the \( e^+e^- \) pairs and the radiation. The radiation spectrum (shown in § 10.2), in both regions, is dominated in the plasma rest frame by US going photons with energy of a few times \( \Gamma m_e c^2 \). In the shock frame the radiation is dominated by DS going photons, beamed into a cone with opening angle \( \sim \Gamma^{-2} \), and a typical energy \( \Gamma^2 m_e c^2 \).

The temperature initially grows in the far US, first exponentially with \( \tau_* \) and slower in the transition region, until it reaches \( T/(m_e c^2) \sim \Gamma \), and then decreases, approximately following this relation.

The transition region ends with a subshock, dissipating a velocity jump \( \delta(\Gamma \beta) \sim 0.1 \) and slightly increasing the temperature to \( 0.4 < T/(m_e c^2) < 0.9 \) for the parameters we calcu-
lated (see Figs. 6 and 12). The immediate DS, following the subshock, is characterized by a small change of velocity, approaching the DS value in $\sim 2$ Thomson optical depths, and a temperature that decreases on a scale of a few Thomson optical depths to $T/(m_e c^2) \sim 0.25$. The ratio of positron density to proton density in the immediate DS reaches a maximum of $\sim 130 \Gamma_u$ (see Fig. 8), approximately when the temperature crosses $T/(m_e c^2) \sim 0.3$, and then decreases. The radiation spectrum in the immediate DS is dominated by a relatively isotropic component with $h\nu \sim 3T$, but a fraction of $10\% - 20\%$ of the energy flux is carried by a high energy photon tail, strongly beamed towards the DS, with a cutoff at $\sim \Gamma_u^2 m_e c^2$ and a nearly flat spectrum $\nu F_\nu \propto \nu^0$ (see Figs. 22 to 24).

- **Optical depths due to Compton scattering and pair production:** In § 10.3 we show that the optical depth of typical photons ($h\nu \sim m_e c^2$) leaving the subshock in the US direction is a few, and is due to both Compton scattering and pair production, the latter having a somewhat smaller effect (see Figs. 27 and 28). Photons with much smaller energies are scattered close to the immediate DS and do not reach the transition region (see Figs. 25 and 26).

Typical DS going photons from the transition region (with energy $\sim \Gamma_u^2 m_e c^2$) undergo very few interactions on the way to the immediate DS (see Figs. 29 to 32).

- **Importance of the $e^+ e^-$ pairs:** In figure 10 we show that the pairs produced along the shock transition and in the immediate DS play an important role in decelerating the US plasma. The energy flux removed from the protons is dominated by pairs over radiation during most of the transition, and the ratio between the two becomes larger as $\Gamma_u$ grows. The pair energy flux is dominated by thermal energy flux, as the temperatures through the transition are relativistic ($T > m_e c^2$).

We find several differences between NR RMS and RRMS: 1. The Thomson optical depth of the transition region is much larger than unity and grows with $\Gamma_u$ in a manner faster than linear. However, the actual optical depth for a typical photon crossing the shock in RRMS (KN corrected as well as pair production) is of order of a few. 2. The temperatures of the pair plasma during the transition become relativistic, $T > m_e c^2$. 3. The relativistic shock has a sonic point crossing, from supersonic flow to subsonic flow, which is found to be a subshock on a scale much shorter than the radiation mean free path. 4. $e^+ e^-$ pairs carry most of the energy and momentum flux during the transition in RRMS, compared to only photons carrying the fluxes in NR RMS. 5. In RRMS a fair fraction of the energy density in the immediate DS is carried by a nonthermal tail of photons, where in NR RMS the radiation is practically always in complete CE with the plasma.

We developed an analytical understanding of the key features in the shock structure and radiation spectrum (section § 11). The main points in the physical picture of the shock are:
• **Immediate DS:** The key to understand the structure and radiation spectrum of RRMS is the conditions in the immediate DS. As we show, the immediate DS of RRMS is close to CPE (see Fig. 38), which, due to fast increase of the number of pairs with temperature, sets the temperature to a large fraction of $m_e c^2$. The CPE conditions also result in a relativistic speed of sound in matter $\beta_{ss} \sim 1/\sqrt{3}$, and combined with the strong radiation drag of velocities that exceed the far DS value $\beta_d \leq 1/3$, a subsonic regime is inevitable. The immediate DS acts as the supplier of photons directed towards the US, which stop the incoming plasma through Compton scattering and pair production.

• **Far US:** The far US features an exponential growth of integral parameters (e.g. $T$, $P_{rad}$) with a characteristic length scale $\lambda_{us}$ [Eq. (97)], derived using a linearized transfer equation. The positron to proton ratio grows with an exponential $\sim 2\lambda_{us}$. The quality of the approximation can be seen in Fig. 36.

• **Deceleration and high energy beam:** We find that the structure of the Lorentz factor when $\Gamma \ll \Gamma_u$ follows $\Gamma(\tau_*) \approx \sqrt{-\tau_*}$ (see Fig. 37), where $\tau_* = 0$ is located at the subshock. This follows mainly from the scaling of the KN cross section and pair production, and from the fact that the optical depth for US going photons is of order few. This approximation follows closely the numerical results up to $\Gamma \approx 1/2\Gamma_u$. Photons that are Compton scattered on the decelerating profile are shown to have a nearly flat, power-law like spectrum with an energy cutoff at $\sim \Gamma_u^2 m_e c^2$. These photons are narrowly beamed in the DS direction. The total optical depth for these photons to reach the immediate DS is very small, and this beam is only stopped further away inside the DS, producing pairs on low energy photons.

• **Length scale of the thermalization length:** The thermalization length is much longer than the shock transition, both in terms of Thomson optical depth and in real distance. The thermalization takes $\sim 10^6$ Thomson optical depths to reach the far DS equilibrium.

• **High energy photon beam:** The high energy beam carrying 10% – 20% of the energy flux in the immediate DS is shown to originate from the transition region (in § 11.3). An approximate expression for the spectral and azimuthal structure of the beam is given in Eq. (108).

Finally, for completeness, we show in § 12 the results of a detailed calculation of the structure of NR RMS including full radiation transport. The results are in agreement with previously published ones, and verify both the numerical methods we use and the diffusion approximation used for solving the problem in previous work.
16. Application to Supernova Shock Breakout

In section §13 we discussed the implications of the results found in §III to the emission of X-rays from a RMS breaking out of a SN. We show in §13.2 that RMSs breaking out of the stellar envelopes of BSGs and WR stars are likely to reach velocities $\beta_u > 0.2$. In §13.3 we show that the radiation expected to be emitted includes a large component of high energy photons coming from the transition region of the RMS. We thus expect that for reasonable stellar parameters, the spectrum emitted during supernovae shock breakouts from BSGs and Wolf-Rayet stars may include a hard component with photon energies reaching tens or even hundreds of keV [see Eq. (32)]. This implies that core-collapse SNe produced by BSGs/WR stars may be searched for by using hard X-ray (wide field) detectors. The detection rate of such events is significantly different than is inferred assuming a thermal emission spectrum (e.g. Calzavara & Matzner 2004), due to both the modification of the intrinsic spectrum and to the reduced absorption of high energy photons.

We have argued in §13.4 that the X-ray outburst XRO080109 associated with SN2008D is most likely due to a fast breakout: The energy, $\sim 10^{49.5}$ erg, duration, $\sim 30$ s, and fast ejecta velocity $\beta_u \sim 1/4$, are all consistent with expected breakout parameters, and the hard X-ray spectrum is a natural consequence of the high velocity, which is independently inferred also from later X-ray and radio observations. Our analysis shows that the spectrum of the X-ray flash XRF060218, associated with SN2006aj, might also be explained as a fast breakout. However, the breakout interpretation of this event is challenged by the long duration of the X-ray flash, and by the high energy, $E \sim 10^{49.5}$ erg, deposited in this explosion in mildly relativistic, $v/c \sim 0.8$, ejecta (see detailed discussion in §13.4).

Wang et al. (2007) have suggested, based on the breakout interpretation of the X-ray outbursts of SN2006aj and SN2008D, that all the low-luminosity Gamma-ray bursts/X-ray flushes associated with SNe, which have smooth light curves and spectra not extending beyond few 100 keV (like those associated with SN1998bw, SN2003lw, SN2006aj), are due to shock breakouts, and do not require the existence of energetic highly relativistic jets. The present analysis, which demonstrates that fast breakouts may indeed produce non-thermal spectra extending to 100’s of keV, supports the viability of the breakout interpretation of low-luminosity Gamma-ray bursts/X-ray flushes associated with SNe. A major challenge for such a scenario is constituted by the requirement of large energy deposition, $E \sim 10^{49.5}$ erg, in the fastest, mildly relativistic ($v/c \sim 0.8$), part of the expanding ejecta.
A. Notations used in part § III

A.1. Subscripts, superscripts and miscellanea

as : Asymptotic upstream behavior
BSG : Blue Super Giant
CE : Compton equilibrium
CPE : Compton-Pair equilibrium
DS : Downstream
d : Asymptotic downstream (postshock) value
dec : Deceleration
diff : Diffusion
eff : Effective
e : Electron value
ee : Electron-electron or positron-positron interaction
ep : Electron-proton interaction
eq : Equilibrium
+- : Electron-positron interaction
KN : Klein Nishina
nl : Non linear
NR : Non-relativistic
p : Proton value
pl : Plasma value
RMS : Radiation Mediated Shock
RRMS : Relativistic Radiation Mediated Shock
rad : Radiation field value
rest : Plasma rest frame value
SN : SuperNova
sh : Shock frame value
US : Upstream
u : Asymptotic upstream (preshock) value
WR : Wolf Rayet
γ : Photon value
γγ : Photon-photon interaction
+: Positron value
^ (hat) : Normalized units
ε : Kinetic energy per proton.
X (dot) : d/dt
A.2. Symbols

\[ a_{BB} = : \text{Radiation constant} \]
\[ F_{rad} (\text{ergs cm}^{-2}\text{s}^{-1}) : \text{Radiation energy flux} \]
\[ P_{rad} (\text{ergs cm}^{-3}) : \text{Radiation momentum flux} \]
\[ h (\text{ergs s}) : \text{Planck’s constant} \]
\[ I(\Omega, \nu) (\text{ergs cm}^{-2}\text{s}^{-1}\text{str}^{-1}\text{Hz}^{-1}) : \text{Specific intensity of radiation field} \]
\[ \eta(\Omega, \nu) (\text{ergs cm}^{-3}\text{s}^{-1}\text{str}^{-1}\text{Hz}^{-1}) : \text{Emissivity coefficient} \]
\[ \ell (\text{cm}) : \text{Photon mean free path} \]
\[ n_e, n_+, n_i, n_{\gamma} (\text{cm}^{-3}) : \text{Number density of electrons, positrons, ions (protons) and photons} \]
\[ n_u (\text{cm}^{-3}) : \text{Upstream proton (and electron) number density} \]
\[ P (\text{ergs cm}^{-3}) : \text{Pressure} \]
\[ P_{rad} (\text{ergs cm}^{-3}) : \text{Radiation pressure} \]
\[ Q (\text{cm}^{-3}\text{s}^{-1}) : \text{Net rate of production} \]
\[ T (\text{erg}) : \text{Electrons & positron temperature} \]
\[ \hat{T} \equiv T/m_e c^2 \]
\[ T^{0z}, T^{zz} (\text{ergs cm}^{-3}) : \text{Components of stress-energy tensor (energy and momentum fluxes, respectively)} \]
\[ x_+ = n_+/n_i : \text{Positron fraction} \]
\[ z (\text{cm}) : \text{Length along flow direction} \]
\[ \chi(\Omega, \nu) (\text{cm}^{-1}) : \text{absorption coefficient} \]
\[ \alpha_e : \text{Fine structure constant} \]
\[ \beta \equiv \sqrt{1 - \Gamma^2} : \text{Flow velocity (units of } c) \]
\[ \Gamma : \text{Flow Lorentz factor} \]
\[ \gamma_{e,th} : \text{Lorentz factor associated with random motion of } e^+ \text{ and } e^- \]
\[ \delta = 1 - \Gamma/\Gamma_u b_u : \text{Asymptotic deceleration parameter} \]
\[ \epsilon_{sc} (\text{ergs}) : \text{Radiation emission cutoff energy due screening} \]
\[ \zeta : \text{Riemann’s zeta function} \]
\[ \eta \equiv \exp(-\gamma_E) = 0.5616 \text{ (where } \gamma_E \approx 0.5772 \text{ is Euler’s constant)} \]
\[ \lambda(\Omega) : \text{Correction factor for bremsstrahlung emission} \]
\[ \lambda_D (\text{cm}) : \text{Debye length } [\equiv \sqrt{T/4\pi e^2(n_e + n_+)}] \]
\[ \Lambda : \text{Logarithmic correction to bremsstrahlung photon production rate} \]
\[ \mu : \text{Cosine of angle relative to positive } z\text{-axis (flow direction)} \]
\[ \nu (\text{Hz}) : \text{Photon frequency} \]
\[ \hat{\nu} \equiv \hbar \nu/m_e c^2 \]
\( \nu_p \equiv m_e c^2 / h \)

\( \sigma \) (cm\(^2\)) : Cross section

\( \sigma_c \) (cm\(^2\)) : Total Compton scattering cross section

\( \sigma_T \) (cm\(^2\)) : Thomson cross section \((8\pi r_0^2/3)\)

\( \tau_\star \) : Optical depth for upstream-going photons

**B. Compton scattering approximation**

We use an approximate Compton Scattering Kernel (CSK) that represents the physically important features of the exact CSK, which demands a lot of computing power to fully implement. The CSK we use is based on these principles:

1. The scattering product is isotropic in the plasma rest frame.

2. The average energy shift of the photon and the energy redistribution function are a function of the temperature \( \hat{T} \) and initial (plasma frame) frequency of the photon \( \hat{\nu} \).

3. The total cross section for scattering is:

\[
\sigma_c(\hat{\nu}, \hat{T}) = \sigma_T \left[ \frac{1+\zeta}{\zeta^3} \left\{ \frac{2(1+\zeta)}{1+2\zeta} - \ln(1+2\zeta) \right\} + \frac{\ln(1+2\zeta)}{2\zeta} - \frac{1+3\zeta}{(1+2\zeta)^2} \right].
\]

(B1)

where \( \zeta \equiv \hat{\nu}(1 + 2\hat{T}) \).

The differential cross section can be written as:

\[
\frac{d\sigma_s}{d\nu d\Omega}(\nu, \Omega \rightarrow \nu', \Omega') = \frac{1}{4\pi} \sigma_c(\nu, T) f_d(\nu, T, \nu'),
\]

(B2)

where \( f_d \) is the spectral redistribution function of the photons, satisfying:

\[
\int_0^\infty f_d(\hat{\nu}, \hat{T}, \hat{\nu}') d\hat{\nu}' = 1
\]

(B3)

\[
\int_0^\infty f_d(\hat{\nu}, \hat{T}, \hat{\nu}') \hat{\nu}' d\hat{\nu}' = \hat{\nu}_0(\hat{\nu}, \hat{T})
\]

(B4)

and \( \nu_0 \) is the average frequency of the photon after the scattering. The prescriptions used for \( \nu_0 \) and \( f_d \) are given below. The treatment is divided to low (NR) temperatures and high (relativistic) temperatures, where the temperature of the transition is \( \hat{T}_m = 0.25 \).

To avoid sharp transitions a smooth interpolation between the two temperature regimes is used, overlapping over a \( \sim 10\% \) interval in \( \hat{T} \). Also, in addition to the scattering terms we prescribe, we make sure that the net change in photon flux is zero, and normalize the scattering accordingly.

This is important since the exact Compton scattering process conserves photon number, and violating this might change the physical results significantly.
B.1. Low T

The Compton scheme for temperatures below $T_m$ was designed to satisfy the NR Compton equilibrium, namely no net energy transfer between the plasma and a photon with $\nu = 4\tilde{T}$ as well as the asymptotic single scattering energy shift of a photon [see, e.g. Rybicki & Lightman (1979)]. For $\nu < 4\tilde{T}$ we use:

$$\frac{\tilde{\nu}}{\nu} = \min \left( \frac{1 + 4\tilde{T}(4\tilde{T} + 1) - \nu(\nu + 1)}{(1 + a_\nu \nu)^3}, \frac{4\tilde{T}}{\nu} \right),$$ (B5)

and for $\nu > 4\tilde{T}$ we use:

$$\frac{\tilde{\nu}}{\nu} = \frac{1}{1 + \log \left( \frac{\nu + 1}{4\tilde{T} + 1} \right)},$$ (B6)

where $a_\nu$ is a function of $T$. We chose the value for this parameter so that for a Wien spectrum of photons at temperature $\tilde{T}$, the energy transfer between photons and electrons of the same $\tilde{T}$ will be zero. Photons with energy less than $4\tilde{T}$ gain energy from the electrons on average and those with energy more than $4\tilde{T}$ lose energy to the electrons. The energy gain and energy loss terms are proportional to:

$$P_{\text{gain}}(\tilde{T}, a) \propto \int_{0}^{4\tilde{T}} d\tilde{\nu} \tilde{\nu}^3 e^{-\tilde{\nu}/\tilde{T}} \sigma_c(\tilde{\nu}, \tilde{T}) \left( \tilde{\nu}_0(\tilde{\nu}, \tilde{T}, a) - \tilde{\nu} \right)$$ (B7)

$$P_{\text{loss}}(\tilde{T}) \propto \int_{4\tilde{T}}^{\infty} d\tilde{\nu} \tilde{\nu}^3 e^{-\tilde{\nu}/\tilde{T}} \sigma_c(\tilde{\nu}, \tilde{T}) \left( \tilde{\nu}_0(\nu, T) - \tilde{\nu} \right).$$ (B8)

We calculate $a_\nu$ so that $P_{\text{gain}} = P_{\text{loss}}$. We find a best fit 4th order polynomial in $\log(\tilde{T})$:

$$a_\nu(\tilde{T}) = -0.003763 \log(\tilde{T})^4 - 0.0231 \log(\tilde{T})^3$$

$$-0.01922 \log(\tilde{T})^2 - 0.129 \log(\tilde{T}) + 3.139.$$ (B9)

We set $a_\nu(\tilde{T} < 0.01) = a_\nu(\tilde{T} = 0.01)$. This interpolation is good to less than a percent.

Photon redistribution The photon distribution follows the shape of a thermal spectrum with a target temperature $\tilde{T}_{\text{tar}} = \tilde{\nu}_0(\nu, \tilde{T})/4$:

$$f_{\tilde{\nu}}(\tilde{\nu}, \tilde{T}, \tilde{\nu}') = A \tilde{\nu}^3 e^{-\tilde{\nu}'/\tilde{T}_{\text{tar}}},$$ (B10)

where

$$A = \left( \int_{0}^{\infty} \tilde{\nu}^3 e^{-\tilde{\nu}'/\til{T}_{\text{tar}}} d\tilde{\nu}' \right)^{-1} = \frac{1}{6\tilde{T}_{\text{tar}}^4}.$$ (B11)
B.2. High T

High \( \nu \) - Klein Nishina

For \( \hat{\nu} > 1/(4\hat{T}) \), in the Klein-Nishina regime, we distribute the photons according to:

\[
f_d(\hat{\nu}, \hat{T}, \hat{\nu}') = \frac{\hat{\nu}'^2 e^{-\hat{\nu}'/\hat{T}} \sigma(\hat{\nu}', \hat{T})}{a_d(T)\hat{T}^3 \sigma_T}.
\] (B12)

The value of \( a_d \) is calculated so that the integral over \( f_d(\hat{\nu}, \hat{T}, \hat{\nu}') d\hat{\nu}' \) is 1. This form ensures that a Wien spectrum with a relativistic temperature will go back to itself if the plasma has the same temperature as well. A best fit 4th order polynomial for \( a_d \) in \( \log(\hat{T}) = \log_e(\hat{T}) \) gives:

\[
a_d(\hat{T}) = -0.004611 \log(\hat{T})^4 + 0.007197 \log(\hat{T})^3 \\
+0.09079 \log(\hat{T})^2 - 0.3166 \log(\hat{T}) + 0.3146
\] (B13)

for \( \hat{T} > 5 \), the approximation \( a(\hat{T}) = a(5)(\hat{T}/5)^{-1.7} \) is used. The relative error of the fit is less than a percent for temperatures below \( m_e c^2 \), and less than 10% everywhere.

Low \( \nu \) - Inverse Compton

In this region we wish to maintain a power law spectrum \( I_\nu \propto \nu^2 \), which characterizes the tail of a Comptonized spectrum, and get the ultra relativistic limit of the energy boost \( 16\hat{T}^2 \). For this purpose, we use a photon redistribution:

\[
f_d(\hat{\nu}, \hat{T}, \hat{\nu}') \propto \sqrt{\nu} e^{-\frac{\sqrt{\nu}}{2\hat{T}^{3/2}}}.
\] (B14)

We use a cutoff at \( 8\hat{T} \) to avoid overtaking of the spectrum at high frequencies, and normalize accordingly.

C. Speed of sound in matter

The calculation of the speed of sound in a mildly relativistic fluid is not trivial, hence we present a short derivation of a general formula for a plasma of electrons, protons and \( e^+ e^- \) pairs, neglecting the thermal pressure of the protons (valid for \( T \ll m_p c^2 \)). Here \( n_l \) is number density of leptons (electrons+positrons), and \( n_p = n_l/(2x_+ + 1) \). For convenience we use \( \hat{p} \equiv p/m_e c^2 \), \( \hat{\epsilon} \equiv \epsilon/m_e c^2 \).

Let \( s \) be the entropy per lepton. Rewriting Eqs. (51) and (52) we have:

\[
\hat{p} = n_i \hat{T} \\
\hat{\epsilon} = n_i(1 + \tilde{f}(\hat{T}) + \frac{m_p}{2x_+ + 1}),
\] (C1)

(\( C2 \))

where we define:

\[
\tilde{f}(\hat{T}) = \frac{3}{2} f(\hat{T}) \hat{T}.
\] (C3)
The derivative with respect to $\hat{T}$ at constant $s$ is denoted by $\cdot'$. Constant entropy per lepton implies:

$$\left(\frac{\dot{e}}{n_l}\right)' = -\dot{p} \left(\frac{1}{n_l}\right)'$$

(C4)

therefore

$$\hat{p}' = \frac{\hat{T}n_l'}{n_l},$$

(C5)

hence:

$$\hat{p}' = n_l \hat{f}' + n_l$$

(C6)

$$\hat{e}' = n_l \hat{f}' + \left(1 + \hat{f} + \frac{m_p}{2x+1}\right) \frac{n_l \hat{f}'}{\hat{T}}.$$  

(C7)

The speed of sound will then be:

$$\beta_{ss} = \frac{c_{ss}}{c} = \sqrt{\frac{\hat{p}'}{\hat{e}'}} = \sqrt{\frac{1 + 1/\hat{f}'}{1 + \hat{f} + \frac{m_p}{2x+1} + \hat{T}}}.$$  

(C8)

This equation can be easily verified to obey the asymptotic NR and ultra relativistic limits.

### D. Transformations and definitions

For completeness, we bring some definitions and transformations that are useful. Transformation of the azimuthal direction $\mu$:

$$\mu_{sh} = \frac{\mu + \beta}{1 + \beta \mu}; \quad \mu = \frac{\mu_{sh} - \beta}{1 - \beta \mu_{sh}}$$  

(D1)

$$\nu_{sh} = \nu \Gamma (1 + \beta \mu); \quad \nu = \nu_{sh} \Gamma (1 - \beta \mu_{sh})$$  

(D2)

$$F_{rad} = \int d\Omega d\nu I_{\nu}(\mu)$$  

(D3)

$$P_{rad} = c^{-1} \int d\Omega \mu^2 d\nu I_{\nu}(\mu).$$  

(D4)

In equations (D3) and (D4) all variables ($F_{rad}$, $P_{rad}$, $\nu$, $I_{\nu}(\mu)$ and $\mu$) are measured in the same frame. We now give the transformation properties of the terms appearing in Eq. (46), the radiation transfer equation:

$$\frac{I(\mu, \nu)}{I_{sh}(\mu_{sh}, \nu_{sh})} = \left(\frac{\nu}{\nu_{sh}}\right)^3$$  

(D5)

$$\frac{\eta(\mu, \nu)}{\eta_{sh}(\mu_{sh}, \nu_{sh})} = \left(\frac{\nu}{\nu_{sh}}\right)^2$$  

(D6)

$$\frac{\chi(\mu, \nu)}{\chi_{sh}(\mu_{sh}, \nu_{sh})} = \left(\frac{\nu}{\nu_{sh}}\right)^{-1}$$  

(D7)
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