A Brief Note on Velocity and Magnitude Correlations

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(Dated: December 14, 2010)

This is a brief explanatory note on the code 'pairV' to compute the magnitude covariance matrix from peculiar miton, and the peculiar velocity two-point function. For further references, see Hui & Greene, Phys. Rev. D73, 123526 (2006) [arXiv:astro-ph/0512159], and Davis et al. (2010) [arXiv:1012.2912].

PACS numbers:

Let us first recall that the aim of supernova cosmology is to fit observations with the following relation:

$$m(z) = 5\log_{10} d_L(z) + M = \frac{5}{\ln 10}\ln d_L(z) + M \quad (1)$$

where m is the apparent magnitude, d_L is the luminosity distance, and z is the redshift. Note that the Hubble constant H_0 can be absorbed into the definition of the zero-point M.

We are interested in the part of the magnitude covariance matrix $\langle \delta m_i \delta m_j \rangle$, where i and j label two supernovae, that comes from peculiar motion; let us call this $C_{ij}^{\rm vel.}$. The code 'pairV' outputs precisely this quantity (known as 'Cv' in the code), as well as the velocity two-point function (known as 'xiV' in the code; see below). The correlation $C_{ij}^{\rm vel.}$ can be expressed in two different ways. From [2], an observer-centric form:

$$C_{ij}^{\rm vel.} = \left[\frac{5}{\ln 10}\right]^2 \left[1 - \frac{a_i}{a_i'}\frac{c}{\chi_i}\right]\left[1 - \frac{a_j}{a_j'}\frac{c}{\chi_j}\right] c^{-2} \quad (2)$$
$$D_i' D_j' \int_0^\infty \frac{dk}{2\pi^2} P(k, a=1)$$
$$\sum_{\ell=0}^{\infty} (2\ell+1) j_\ell'(k\chi_i) j_\ell'(k\chi_j) P_\ell(\hat{\mathbf{x}}_{\mathbf{i}} \cdot \hat{\mathbf{x}}_{\mathbf{j}})$$

and from [3, 4], a separation-centric form:

$$C_{ij}^{\rm vel.} = \left[\frac{5}{\ln 10}\right]^2 \left[1 - \frac{a_i}{a_i'}\frac{c}{\chi_i}\right]\left[1 - \frac{a_j}{a_j'}\frac{c}{\chi_j}\right] c^{-2} \quad (3)$$
$$[\,(\hat{\mathbf{x}}_{\mathbf{i}} \cdot \hat{\mathbf{r}})(\hat{\mathbf{x}}_{\mathbf{j}} \cdot \hat{\mathbf{r}})\Pi(r) + [\hat{\mathbf{x}}_{\mathbf{i}} \cdot \hat{\mathbf{x}}_{\mathbf{j}} - (\hat{\mathbf{x}}_{\mathbf{i}} \cdot \hat{\mathbf{r}})(\hat{\mathbf{x}}_{\mathbf{j}} \cdot \hat{\mathbf{r}})]\Sigma(r)\,]$$
$$\Pi(r) \equiv D_i' D_j' \int_0^\infty \frac{dk}{2\pi^2} P(k, a=1)\left[j_0(kr) - \frac{2 j_1(kr)}{kr}\right]$$
$$\Sigma(r) \equiv D_i' D_j' \int_0^\infty \frac{dk}{2\pi^2} P(k, a=1)\frac{j_1(kr)}{kr}$$

where j_ℓ is the spherical Bessel function, j_ℓ' is its derivative with respect to its argument (useful: $j_\ell' = j_{\ell-1} - (\ell+1) j_\ell / x$), P_ℓ is the Legendre polynomial, and $\hat{\mathbf{x}}_{\mathbf{i}}$ and $\hat{\mathbf{x}}_{\mathbf{j}}$ represent the unit vectors pointing towards SNe i and j, r is the comoving separation between the two SNe, and $\hat{\mathbf{r}}$ is the unit vector pointing along the separation. That eq. (2) and (3) are equivalent can be shown as follows. The code 'pairV' uses eq. (2) when the parameter 'imethod' is set to 1, and uses eq. (3) when 'imethod' is set to 2. The latter method is the faster one. On the other hand, the former method provides a more direct link to the observed velocity angular power spectrum (eq. [9]). Consistency between the two is a good check.

It comes down to evaluating the two point velocity correlation $\xi_{12}^{\rm vel.}$

$$C_{12}^{\rm vel.} = \left[\frac{5}{\ln 10}\right]^2 \left[1 - \frac{a_1}{a_1'}\frac{c}{\chi_1}\right]\left[1 - \frac{a_2}{a_2'}\frac{c}{\chi_2}\right]\frac{\xi_{12}^{\rm vel.}}{c^2} \quad (4)$$

where 1 and 2 label the two SNe in question, and

$$\xi_{12}^{\rm vel.} \equiv \langle (\mathbf{v_1} \cdot \hat{\mathbf{x}}_{\mathbf{1}})(\mathbf{v_2} \cdot \hat{\mathbf{x}}_{\mathbf{2}}) \rangle \quad (5)$$

The velocity correlation function $\xi_{12}^{\rm vel.}$ is precisely the quantity 'xiV' computed in the code. This is really an old subject (see e.g. [4]). The main reason we go over the derivation here is that minor errors have appeared in some recent literature, as pointed out by [3] (see [5, 6]). It is also useful to see how two completely different looking expressions, i.e. eq. (2) and (3), are actually equivalent.

Using linear theory, it can be shown that

$$\xi_{12}^{\rm vel.} = D_1' D_2' \int \frac{d^3k}{(2\pi)^3} k^{-2} P(k, a=1) \quad (6)$$
$$(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}_{\mathbf{1}})(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}_{\mathbf{2}}) e^{-i\mathbf{k}\cdot(\mathbf{x_1} - \mathbf{x_2})}$$

where $\mathbf{x_1}$ and $\mathbf{x_2}$ are the comoving positions of the two SNe in question, $\hat{\mathbf{n}}_{\mathbf{1}}$ and $\hat{\mathbf{n}}_{\mathbf{2}}$ are the unit vectors pointing in these directions, $P(k, a=1)$ is the mass power spectrum today, and D_1' and D_2' are the derivatives of the growth factor with respect to conformal time at the two redshifts of interest.

An observer-centric approach is to use

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}_{\mathbf{2}} e^{i\mathbf{k}\cdot x_2} = 4\pi \sum_{\ell,m} i^{\ell-1} j_\ell'(k\chi_2) Y_{\ell m}^*(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{x}}_{\mathbf{2}}) \quad (7)$$

where j_ℓ is the spherical Bessel function, j_ℓ' is its derivative (with respect to its argument, not conformal time), and $Y_{\ell m}$'s are the spherical harmonics. Performing the integral over $\hat{\mathbf{k}}$ in eq. (6), and using

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$\int d\Omega_k Y^*_{\ell m}(\hat{\mathbf{k}}) Y_{\ell' m'}(\hat{\mathbf{k}}) = \delta_{\ell\ell'}\delta_{mm'}$ and $P_\ell(\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{x}}_\mathbf{2}) = 4\pi/(2\ell+1)\sum_m Y^*_{\ell m}(\hat{\mathbf{x}}_\mathbf{1}) Y_{\ell m}(\hat{\mathbf{x}}_\mathbf{2})$, it is straightforward to show that

$$\xi^{\rm vel.}_{12} = D'_1 D'_2 \int \frac{dk}{2\pi^2} P(k, a=1) \sum_\ell (2\ell+1) j'_\ell(k\chi_1) j'_\ell(k\chi_2) P_\ell(\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{x}}_\mathbf{2}) \tag{8}$$

from which eq. (2) can be obtained (see [1] for details). The above expression is observer-centric in the sense that one can easily read off from it the angular velocity power spectrum as seen by the observer:

$$C^{\rm vel.}_\ell = D'_1 D'_2 \int \frac{2dk}{\pi} P(k, a=1) j'_\ell(k\chi_1) j'_\ell(k\chi_2) \tag{9}$$

Here, 1 and 2 can refer to the same redshift, or two different redshifts.

A different approach to reducing eq. (6) is to first note that by symmetry arguments [4]:

$$\langle v_i(\mathbf{x_1}) v_j(\mathbf{x_2})\rangle = [\Pi(r) - \Sigma(r)]\hat{r}_i \hat{r}_j + \Sigma(r)\delta_{ij} \tag{10}$$

where i and j here, unlike in the rest of the paper, label the spatial directions rather than the SNe, r is the comving separation between points 1 and 2, and $\hat{\mathbf{r}}$ is the associated unit vector. Suppose $\hat{\mathbf{r}}$ points in the z direction, then the above matrix is diagonal, with diagonal entries Σ, Σ, Π i.e. Σ is the perpendicular velocity correlation and Π is the parallel velocity correlation. Here, parallel and perpendicular are defined by the separation vector between the two SNe (hence a separation-centric approach). From this matrix, one can deduce that

$$\xi^{\rm vel.}_{12} = (\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{r}})(\hat{\mathbf{x}}_\mathbf{2} \cdot \hat{\mathbf{r}})\Pi(r) + [\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{x}}_\mathbf{2} - (\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{r}})(\hat{\mathbf{x}}_\mathbf{2} \cdot \hat{\mathbf{r}})]\Sigma(r) \tag{11}$$

where $[\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{x}}_\mathbf{2} - (\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{r}})(\hat{\mathbf{x}}_\mathbf{2} \cdot \hat{\mathbf{r}})]$ can be written as $\sin\theta_1 \sin\theta_2$ if $\hat{\mathbf{x}}_\mathbf{1} \cdot \hat{\mathbf{r}} = \cos\theta_1$ and $\hat{\mathbf{x}}_\mathbf{2} \cdot \hat{\mathbf{r}} = \cos\theta_2$. Comparing this expression with eq. (6), one can see that

$$\Pi(r) = D'_1 D'_2 \int \frac{d^3k}{(2\pi)^3} k^{-2} P(k, a=1) (\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})^2 e^{i\mathbf{k}\cdot\mathbf{r}} \tag{12}$$

Using

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \tag{13}$$

and integrating over $\hat{\mathbf{k}}$ (choosing $\hat{\mathbf{r}}$ to lie in the z direction for instance), one can see that only $\ell = 2$ and $\ell = 0$ survives. Finally, using $j_2 = 3j_1/x - j_0$, one obtains:

$$\Pi(r) = D'_1 D'_2 \int \frac{dk}{2\pi^2} P(k, a=1) \left[j_0(kr) - \frac{2j_1(kr)}{kr} \right] \tag{14}$$

The perpendicular counterpart can be similarly obtained from

$$\Sigma(r) = D'_1 D'_2 \int \frac{d^3k}{(2\pi)^3} \frac{P(k, a=1)}{k^2} (\hat{\mathbf{k}} \cdot \hat{\mathbf{x}})^2 e^{i\mathbf{k}\cdot\mathbf{r}} \tag{15}$$

with $\hat{\mathbf{x}}$ pointing in the x direction while $\hat{\mathbf{r}}$ points in the z direction. A few manipulations yield

$$\Sigma(r) = D'_1 D'_2 \int \frac{dk}{2\pi^2} P(k, a=1) \frac{j_1(kr)}{kr} \tag{16}$$

reproducing the results of [4], and giving our eq. (3).

Thanks are due to Josh Frieman for discussions. The work is supported in part by DOE and NASA.

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[2] Combine eq. 22, D7 and D10 of [1], setting in eq. 22 the survey geometry to be two points centered at the two SNe in question.

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