Physics 8048
Take home final, due 12/15/14
Lam Hui

Please work on this take home final completely on your own. You are welcome to consult the class lecture notes, Srednicki and Zee, but nothing else. Please hand it in by noon on 12/15/14 at my mailbox on the 7th floor.

1. In class, we discussed the case of SSB of SU(N) gauge symmetry (SU(5) being a concrete example), in the case of a Higgs sector that transforms in the adjoint representation. We made the claim that the Higgs in this case can be represented as an $N \times N$ (traceless, Hermitian) matrix, as opposed to a column vector with $N^2 - 1$ components. Here, I want you to show that the two descriptions are indeed equivalent – we went over this briefly in class already, but I want to make sure you got it.

Let us start with the column vector description first (which is what we are used to, if the Higgs is in the fundamental representation). Consider a set of real scalar fields φ^a, $a = 1, 2, ..., N^2 - 1$, that transform under the adjoint representation of $SU(N)$. The covariant derivative – in the notation we have been adopting in class – is written as

$$D_\mu \varphi^b = \partial_\mu \varphi^b - igA^a_\mu [T^a]^{bc} \varphi^c \,, \tag{1}$$

where $[T^a]^{bc}$ represents the generators in the adjoint representation i.e. each $[T^a]$ is a $(N^2 - 1) \times (N^2 - 1)$ matrix. We know the actual numerical values of each component of the matrix takes the form: $[T^a]^{bc} = -if^{abc}$, where f^{abc}'s are the structure constants.

Now, consider an alternative description of this system: arrange the φ^a's into an $N \times N$ traceless unitary matrix by defining $\Phi = \varphi^a T^a$. In other words, here, each T^a is an $N \times N$ (traceless, Hermitian) matrix i.e. they are the generators in the *fundamental* representation (even though we continue to think of the φ^a's, or more abstractly Φ, as in the adjoint representation). Thus, take care to distinguish between T^a used here from the $[T^a]$ used above. *Show that* Eq. (1) can be re-expressed in this language as:

$$D_\mu \Phi = \partial_\mu \Phi - igA^a_\mu [T^a, \Phi] \tag{2}$$

2. This is related to the previous problem, but focuses on the case of SU(5). You can find discussion of this case in Chapter 84 of Srednicki. Once again, we have Higgs in the adjoint representation, and using the notation in problem 1, we choose the Higgs VEV to be $\langle \Phi \rangle = v \times \text{diag.}(-1/3, -1/3, -1/3, 1/2, 1/2)$ i.e. a matrix with non-zero entries only along the diagonal (here v is some constant). We have the task then to figure out which SU(5) generators are unbroken. Recall that all SU(5) generators T^a's (in the fundamental representation) can in general be written as a linear combination of 3 different types: (1) matrices with a 1 in the upper triangular part and a corresponding 1 in the lower triangular part; (2) matrices with an $-i$ in the upper triangular part and a correspoinding i in the lower triangular part; (3) matrices with non-zero entries only along the diagonal, of the form diag. $(1, ..., 1, -n, 0, ...)$ where there are n number of 1's compensated by a $-n$ to make the matrix tracefree; for SU(5), n can be 1, 2, 3, or 4. (In all three categories, the matrices need to be suitably normalized of course.) We argued in class that the unbroken generators form a SU(3) × SU(2) × U(1) sub-algebra, and we say the generator for U(1) can be identified $T^a \propto \langle \Phi \rangle$ because this obviously commutes with $\langle \Phi \rangle$. Such a T^a can be obtained from a suitable linear combination of the matrices in category (3): let me call this combination T^{24} following Srednicki. One of you asked: what about the other matrices in category (3)? I don't think I gave you the correct answer in the class. You can check explicitly that they also commute with $\langle \Phi \rangle$ and thus should belong to the set of unbroken generators. So, here's my question for you: there are 4 linearly independent generators of the category 3; one of them is T^{24} itself; the other three must somehow be generators of SU(3) × SU(2); can you show it?

3. Srednicki problem 88.3.

4. Srednicki problem 91.1.

5. In this problem, you are asked to work out a particular example of effective field theory. This is a general idea that, at low energies, the physical implications of a higher energy can be captured by an effective theory – this was the whole idea of integrating out the UV modes that we discussed last semester. Here, I ask you to work out a particular instance of effective field theory in weak interactions.

Srednicki equation 88.23 tells us the complete gauge-lepton interactions present in the standard model (after spontaneous symmetry breaking). *Work out* the tree scattering amplitude for the process $\nu_\mu + e^- \to \nu_e + \mu^-$ using the interaction vertices present in that equation (but summed over generations of course). To be definite, let us call the momentum of $e^-, \nu_\mu, \nu_e, \mu^-$: p_1, p_2, p'_1, p'_2. Then, take the limit of $(p_1 - p'_1)^2$ much less than the gauge boson masses (except photons of course), and *show that* you can obtain the same answer from the effective theory shown in Srednicki equation 88.30 (again, with a suitable sum over generations). Notice how in this effective theory, the massive gauge bosons disappear: the only interactions are these vertices that involve four fermions. This was Fermi's theory. From our point of view, the massive gauge bosons (UV degrees of freedom) can be integrated out of the standard model to address low energy questions. From Fermi's point of view, he didn't even have to know about the existence of massive gauge bosons; his 4-fermion theory is sufficient to give accurate descriptions of low energy scattering. His theory breaks down when the energy approaches the mass of W's and Z. In other words, the mass scale represents the cut-off of Fermi's effective theory. Our modern viewpoint is that, except in those rare cases of UV complete theories, almost any theory we use has some cut-off which tells us our theory has a limited range of validity. There's no shame in using such theories – in fact, we do this all the time without thinking about it e.g. we use Newton's theory to predict how rockets fly – they give good accurate predictions as long as we stay below the cut-off.