Physics 8048
Problem Set 6, due 11/19/14
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This is a problem set on representation theory. Please read Srednicki Chapter 70. There are a number of quantities defined there that we did not discuss in the lecture.

1. Srednicki problem 70.1. Here, $T(R)$ is the normalization as defined in Srednicki eq. 70.9, $C(R)$ is the quadratic Casimir $T^a_R T^a_R$, $D(A)$ is the dimension of the adjoint representation, while $D(R)$ is the dimension of the representation R.

2. Srednicki problem 70.2, parts a,b,c. Comments: for SU(N), Srednicki chooses the normalization of the generators in the fundamental representation such that $T(N) = 1/2$, where the "N" of $T(N)$ refers to the fundamental representation. Once this normalization is fixed, the normalization of other representations, such as the adjoint or representations by direct products of the fundamental and its conjugate, is fixed also – and their corresponding T's are not necessarily 1/2.

3. Srednicki problem 70.6. Comments: the reason for claiming $(F_{\mu\nu})^a$ transforms as in the adjoint is because of the a index: the number of possible F^a's is the same as the number of generators, which is also the dimension of the adjoint representation. This result to be derived is called the Bianchi identity, for it is a direct analog of the same so named in general relativity. What does the Bianchi identity reduce to, in the case of U(1) i.e. the Maxwell field strength $F_{\mu\nu}$?

4. For SU(N), we discuss the set of states labeled by ${\varphi_i}^j$, where i and j each range from 1 up to N and are indices of the fundamental (N) and its conjugate (N^*). Moreover, ${\varphi_i}^i = 0$ (i.e. traceless). There are thus $N^2 - 1$ elements for ${\varphi_i}^j$, based on which we claimed they belong to the adjoint representation. *Show that* this is indeed the case, i.e. they transform in a way that is equivalent to the adjoint representation. Here's how you might proceed: recall ${\varphi_i}^j$ transforms according to ${\varphi_i}^j \to {\varphi'_i}^j = {U_i}^m {\varphi_m}^n {(U^\dagger)_n}^j$, where ${U_i}^m \sim {\delta_i}^m - i\theta^a {(T^a)_i}^j$ with ${(T^a)_i}^j$ being the generators in the fundamental representation N; by writing ${\varphi_i}^j$ as ${\varphi_i}^j = \varphi^a {(T^a)_i}^j$, show that φ^a is a set of $N^2 - 1$ objects that transform exactly as in the adjoint representation. Comment: the mapping from the set of objects ${\varphi_i}^j$ to the set of objects φ^a is precisely part of what is involved in a similarity transformation, i.e. abstractly think of ${(T^a)_i}^j$ as an $(N^2 - 1) \times (N^2 - 1)$ matrix with one index labeled by a, and the other index labeled by the pair ${}_i{}^j$.

5. a. Show that $A(R) = 0$ for the fundamental representation of SU(2). Recall the fundamental rep. of SU(N) was defined in chapter 24 (p. 148), which (up to a normalization factor) coincides with the Pauli matrices for SU(2). $A(R)$ is the anomaly coefficient defined in chapter 70. If one wants to have $\mathrm{Tr.}(T^a_R T^b_R) = T(R)\delta^{ab}$ with $T(R) = 1/2$, what normalization factor should you use for each of these Pauli matrices?

b. The fundamental rep. of SU(3) according to the same definition would give $T^a = \mathcal{N}\lambda^a$:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (1)$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

known as the Gell-Mann matrices. What normalization factor $\mathcal{N}$ should be chosen to give $T(R) = 1/2$? Work out the structure constant from this fundamental representation i.e. gives the numerical values for all non-vanishing components of f^{abc}. Work out C(R) and T(A), where R is the fundamental rep as defined here, and A is the adjoint. Check that A(R) $\neq$ 0. It is defined to be 1. A side comment: note how T^1, T^2, T^3, T^8 form a sub-algebra of SU(2) x U(1).