Physics 8048
Problem Set 5, due 11/5/14
Lam Hui

1. Starting from the expression for V^μ in Srednicki eq. 63.3, show that $F_2(0) = \alpha/(2\pi)$ as in eq. 63.22. The derivation is of course given in Srednicki, but see if you can reproduce the result yourself. This is the key result that gives us the loop correction to the magnetic moment of the electron.

2. Srednicki problem 68.1

3. Srednicki problem 69.2. This result motivates the definition of the quadratic Casimir in chapter 70.

4. In this problem, let's return to the subject of QED, and prove $Z_1 = Z_2$ i.e. gauge invariance is preserved by renormalization. (You might wonder: what does that even mean if we have to gauge fix anyway? You will find out below.) This is treated in Srednicki chapter 68, but I would like to clarify a few points.

a. Recall our Lagrangian:

$$\mathcal{L} = iZ_2\bar{\Psi}\gamma^\mu\partial_\mu\Psi - Z_m m\bar{\Psi}\Psi - \frac{1}{4}Z_3F^2 + Z_1 e\bar{\Psi}\gamma^\mu A_\mu\Psi - \frac{1}{2\xi}(\partial\cdot A)^2 \tag{1}$$

Recall the global U(1) symmetry: $\Psi \to e^{-ie}\Psi$ i.e. $\delta\Psi = -ie\Psi$. Let's run the Noether argument: consider a slightly deformed, infinitesimal version of the symmetry transformation: $\Psi \to \Psi + \epsilon\rho\delta\Psi$, where ϵ is just a small number to keep track of order, and ρ is some general function of x. The fact that we have a U(1) symmetry tells us the action must change under the deformed transformation by $\delta S = \int d^4x\epsilon J_N^\mu\partial_\mu\rho$. *Show that* the Noether current is $J_N^\mu = eZ_2\bar{\Psi}\gamma^\mu\Psi$. The Ward identity (problem set 5) tells us:

$$\frac{\partial}{\partial x_0^\mu}\langle J_N^\mu(x_0)\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle = -e\langle\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle\left[\delta(x_0 - x_1) - \delta(x_0 - x_2)\right] \tag{2}$$

On the other hand, the A^μ equation of motion takes the form:

$$\mathcal{D}^\mu{}_\nu A^\nu = \frac{Z_1}{Z_2}J_N^\mu \quad , \quad \mathcal{D}^\mu{}_\nu \equiv -Z_3\delta^\mu{}_\nu\Box + Z_3\partial^\mu\partial_\nu - \frac{1}{\xi}\partial^\mu\partial_\nu \tag{3}$$

Substituting this in the Ward identity implies

$$\frac{Z_2}{Z_1}\frac{\partial}{\partial x_0^\mu}D^\mu{}_\nu(x_0)\langle A^\nu(x_0)\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle = -e\langle\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle\left[\delta(x_0 - x_1) - \delta(x_0 - x_2)\right] \tag{4}$$

Question: why is there no extra contact term associated with the use of the equation of motion in the correlation function?

Next, let's recall the relationship between the (full) photon-fermion-fermion three-point function $\langle A^\nu\Psi_\alpha\bar{\Psi}_\beta\rangle$ and the (full) cubic vertex function $iV^\nu{}_{\alpha\beta}$: the latter is simply the former with the full external propagators removed. The relation in real space reads:

$$\begin{aligned}\langle A^\nu(x_0)\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle = \quad & \int d^4y_0 d^4y_1 d^4y_2 \tfrac{1}{i}\Delta^\nu{}_{\nu'}(x_0 - y_0)\tfrac{1}{i}S_{\alpha\alpha'}(x_1 - y_1) \\ & iV^{\nu'}{}_{\alpha'\beta'}(y_0, y_1, y_2)\tfrac{1}{i}S_{\beta'\beta}(y_2 - x_2)\end{aligned} \tag{5}$$

where Δ/i and S/i are the full photon and fermion two-point functions respectively. Substituting this into eq. (7), we see that we have a term: $\partial_{x_0^\mu} D^\mu{}_\nu(x_0)\Delta^\nu{}_{\nu'}(x_0 - y_0)/i$. *Show that* it gives

$$\frac{\partial}{\partial x_0^\mu} D^\mu{}_\nu(x_0)\langle A^\nu(x_0)A^{\nu'}(y_0)\rangle = -i\frac{\partial}{\partial x_0^{\nu'}}\delta(x_0 - y_0)\,. \tag{6}$$

In doing so, you will be using Ward identity and the A^ν equation of motion yet again; be careful about when you have or have not contact terms. Putting everything together, we thus have

$$\begin{aligned}\tfrac{Z_2}{Z_1}\textstyle\int d^4y_1 d^4y_2 \tfrac{1}{i}S_{\alpha\alpha'}(x_1 - y_1)\tfrac{\partial}{\partial x_0^{\nu'}}V^{\nu'}{}_{\alpha'\beta'}(x_0, y_1, y_2)\tfrac{1}{i}S_{\beta'\beta}(y_2 - x_2)& \\ = -e\tfrac{1}{i}S_{\alpha\beta}(x_1 - x_2)\left[\delta(x_0 - x_1) - \delta(x_0 - x_2)\right]\,.&\end{aligned} \tag{7}$$

What we have effectively shown is that the left hand side of the Ward identity is related to the cubic vertex function by:

$$\frac{\partial}{\partial x_0^{\nu'}}\langle J_N^{\nu'}(x_0)\Psi_\alpha(x_1)\bar\Psi_\beta(x_2)\rangle = \frac{\partial}{\partial x_0^{\nu'}}\left[\frac{Z_2}{Z_1}\int d^4y_1 d^4y_2 \frac{1}{i}S_{\alpha\alpha'}(x_1 - y_1)V^{\nu'}{}_{\alpha'\beta'}(x_0, y_1, y_2)\frac{1}{i}S_{\beta'\beta}(y_2 - x_2)\right], \tag{8}$$

which is consistent with Srednicki's starting point in his eqs. 68.1 and 68.2; however, his version has the overall derivative removed, which does not seem to be correct – if you rerun the above arguments with the overall derivative removed, I believe you should get extra terms.

Next, using

$$\begin{aligned}\int d^4x_1 iS^{-1}_{\alpha''\alpha}(z_1 - x_1)\frac{1}{i}S_{\alpha\alpha'}(x_1 - y_1) &= \delta_{\alpha''\alpha'}\delta(z_1 - y_1) \\ \int d^4x_2 \frac{1}{i}S_{\beta'\beta}(y_2 - x_2)iS^{-1}_{\beta\beta''}(x_2 - z_2) &= \delta_{\beta'\beta''}\delta(y_2 - z_2)\,,\end{aligned} \tag{9}$$

show that

$$\frac{Z_2}{Z_1}\frac{\partial}{\partial x_0^{\nu'}}V^{\nu'}{}_{\alpha''\beta''}(x_0, z_1, z_2) = -e\,iS^{-1}_{\alpha''\beta''}(z_1 - z_2)\left[\delta(x_0 - z_2) - \delta(x_0 - z_1)\right] \tag{10}$$

Lastly, let's go to Fourier space using the following convention:

$$(2\pi)^4\delta(p + q - p')\tilde V^{\nu'}{}_{\alpha''\beta''}(q, p', p) = \int d^4x_0 d^4z_1 d^4z_2 V^{\nu'}{}_{\alpha''\beta''}(x_0, z_1, z_2)e^{iqx_0 - ip'z_1 + ipz_2} \tag{11}$$

the signs of momenta chosen to match Srednicki's i.e. p and q are ingoing while p' is outgoing. The inverse (full) propagator in Fourier space is defined by

$$S^{-1}_{\alpha''\beta''}(z_1 - z_2) = \int \frac{d^4k}{(2\pi)^4}\tilde S(k)^{-1}e^{ik(z_1 - z_2)} \tag{12}$$

Show that

$$q_{\nu'}\tilde V^{\nu'}{}_{\alpha''\beta''}(q, p', p) = \frac{Z_1}{Z_2}e\left(\tilde S(p')^{-1} - \tilde S(p)^{-1}\right) \tag{13}$$

which is consistent with Srednicki's eq. 68.12. (I hope you appreciate the fact that we got even the sign right, since there were a lot of ways we could have got it wrong.) This is often referred to as the Ward-Takahashi identity, and is the original formulation of the identity. This is a powerful result because it is non-perturbative: $\tilde{V}$ and $\tilde{S}$ are the full vertex and propagator including all loops. Note also that q, p', p need not be on-shell (though they obey conservation $p + q = p'$). A corollary of this identity is that $q_{\nu'} \tilde{V}^{\nu'}{}_{\alpha''\beta''} = 0$ in the limit $q \to 0$.

Eq. (13) tells us that whatever renormalization scheme we use to render $\tilde{V}$ and $\tilde{S}$ finite, it must be such that Z_1/Z_2 is finite. In the $\overline{MS}$ scheme, Z_1 and Z_2 take the form of $1 + O(1/d-4) + ...$ For Z_1/Z_2 to come out finite, it must be true that the poles in Z_1 and Z_2 take exactly the same amplitude i.e. $Z_1 = Z_2$.

Furthermore, let's consider eq. (13) with momenta close to on-shell, and with q taken to be small but non-vanishing. On the right hand side, we expect $\tilde{S}(p)^{-1} \to \gamma^\mu p_\mu + m_{\rm phys.}$ as we approach on-shell-ness. Thus, we find on the right hand side $(Z_1/Z_2)e\gamma^\mu q_\mu$. The left hand side is already order q, so we can take it to be $q_\mu \tilde{V}^\mu(0, p, p)$, where I have suppressed the α'', β'' spinor indices. Comparing the two sides, we thus find $\tilde{V}^\mu(0, p, p) = (Z_1/Z_2)e\gamma^\mu$. Finally, let us recall we define the physical charge $e_{\rm phys.}$ by $\bar{u}(p')\tilde{V}^\mu u(p) = e_{\rm phys.}\bar{u}(p')\gamma^\mu u(p)$, where the momenta are exactly on-shell: $p^2 = p'^2 = -m^2_{\rm phys.}$ and $q^2 = 0$. Thus, sandwiching our relation by $\bar{u}$ and u, we obtain

$$e_{\rm phys.} = \frac{Z_1}{Z_2} e \tag{14}$$

For the $\overline{MS}$ renormalization scheme, we have already argued $Z_1 = Z_2$. Eq. (14) further tells us $e_{\rm phys.} = e$. On the other hand, if we had used the on-shell renormalization scheme, where e is chosen to be equal to $e_{\rm phys.}$, eq. (14) tells us $Z_1 = Z_2$ also holds for the on-shell scheme.

The significance of $Z_1 = Z_2$ is that the QED Lagrangian in eq. (1), without the gauge-fixing term, is invarant under $\Psi \to e^{-ie\lambda}\Psi$, $A_\mu \to A_\mu - \partial_\mu \lambda$, which is the same as the symmetry at tree level. In other words, gauge invariance is preserved by renormalization. Its breaking is confined to the gauge-fixing term. No further breaking terms is introduced by loops.

b. *This part has no problems you need to solve. It is pure notes.* There is a slicker way to see that gauge invariance is preserved at the loop level, without worrying about all these contact terms. It does require a bit of formalism though. It is the formalism of the quantum action, which was covered in chapter 21 but we haven't used it very much. Here's a chance to see how it is useful.

Formalism. Recall that the quantum action Γ is related to the generating function for connected diagrams iW ($Z = e^{iW}$) by the Legendre transform

$$W(J) = \Gamma(\phi) + \sum_a \int d^4x\, \phi^a(x) J^a(x) \tag{15}$$

where ϕ^a can represent any field e.g. the fermion spinor, or the photon, etc. The summation over a sums over all these fields, each with its own source J^a. As with any Legendre transform, we have

$$\frac{\delta W}{\delta J^a(x)} = \phi^a(x) \quad , \quad \frac{\delta \Gamma}{\delta \phi^a(x)} = -J^a(x) \tag{16}$$

with the understanding that one can solve the first equation for J as a function of ϕ and obtain $\Gamma(\phi)$ from $W(J)$, or vice versa for the second equation. Note that if I were to be careful about the possibility of Grassmann variables, I would think of the J derivatives as coming from the right, and the ϕ derivatives as coming from the left. Let's not worry about this too much, but all my expressions are correct if you think of the derivatives this way.

Consider Taylor expanding $\delta W/\delta J$ around $J = 0$:

$$\begin{aligned}\phi^a(x) = \tfrac{\delta W}{\delta J^a(x)} = \tfrac{\delta W}{\delta J^a(x)}\Big|_{J=0} + \textstyle\sum_b \int d^4y \tfrac{\delta^2 W}{\delta J^b(y)\delta J^a(x)}\Big|_{J=0} J^b(y) + \\ \tfrac{1}{2}\textstyle\sum_{b,c}\int d^4y d^4z \tfrac{\delta^3 W}{\delta J^c(z)\delta J^b(y)\delta J^a(x)}\Big|_{J=0} J^c(z)J^b(y) + \dots \,.\end{aligned} \tag{17}$$

To simplify our notation, we will suppress x, y, z, and simply use a, b, c to denote both the kind of fields we have and their respective positions. Furthermore let's write:

$$W^a \equiv \frac{\delta W}{\delta J^a} \quad , \quad W_0^a \equiv \frac{\delta W}{\delta J^a}\Big|_{J=0} \quad , \quad W_0^{ab} \equiv \frac{\delta^2 W}{\delta J^b \delta J^a}\Big|_{J=0} \quad , \quad W_0^{abc} \equiv \frac{\delta^3 W}{\delta J^c(z)\delta J^b(y)\delta J^a(x)}\Big|_{J=0} \tag{18}$$

Each of these derivatives evaluated at $J = 0$ represents a connected correlation function (in the absence of external sources) i.e.

$$\begin{aligned}\langle\phi^a\rangle = \tfrac{\delta iW}{\delta iJ^a}\Big|_{J=0} = W_0^a \quad , \quad \langle\phi^a\phi^b\rangle_c = \tfrac{\delta^2 iW}{\delta iJ^b \delta iJ^a}\Big|_{J=0} = \tfrac{1}{i}W_0^{ab} \\ \langle\phi^a\phi^b\phi^c\rangle_c = \tfrac{\delta^3 iW}{\delta iJ^c\delta iJ^b\delta iJ^a}\Big|_{J=0} = -W_0^{abc} \,.\end{aligned} \tag{19}$$

Note that I have not set the one-point function $\langle\phi^a\rangle$ to zero, which is normally what we do. This is to keep the formalism sufficiently general to include cases of spontaneous symmetry breaking that we will deal with later in the course. Eq. (17) can be rewritten as

$$\phi^a - \langle\phi^a\rangle = W_0^{ab}J^b + \frac{1}{2}W_0^{abc}J^cJ^b + \dots \tag{20}$$

One can solve this iteratively for J as a function of ϕ:

$$J^a = W_0^{-1aa'}[\phi^{a'} - \langle\phi^{a'}\rangle] - \frac{1}{2}W_0^{-1aa'}W_0^{a'b'c'}W_0^{-1c'c}[\phi^c - \langle\phi^c\rangle]W_0^{-1b'b}[\phi^b - \langle\phi^b\rangle] + \dots \tag{21}$$

On the other hand, we can also Taylor expand $\delta\Gamma/\delta\phi$ around $\phi = \langle\phi\rangle$:

$$-J^a = \Gamma^a = \frac{\delta\Gamma}{\delta\phi^a} = \Gamma_0^a + [\phi^b - \langle\phi^b\rangle]\Gamma_0^{ab} + \frac{1}{2}[\phi^b - \langle\phi^b\rangle][\phi^c - \langle\phi^c\rangle]\Gamma_0^{abc} + \dots \,, \tag{22}$$

where we have adopted the notation

$$\Gamma^a \equiv \frac{\delta\Gamma}{\delta\phi^a} \quad , \quad \Gamma_0^a \equiv \frac{\delta\Gamma}{\delta\phi^a}\Big|_{\phi=\langle\phi\rangle} \quad , \quad \Gamma_0^{ab} \equiv \frac{\delta^2\Gamma}{\delta\phi^b\delta\phi^a}\Big|_{\phi=\langle\phi\rangle} \quad , \quad \Gamma_0^{abc} \equiv \frac{\delta^3\Gamma}{\delta\phi^c\delta\phi^b\delta\phi^a}\Big|_{\phi=\langle\phi\rangle} + \dots \,, \tag{23}$$

Note that $\Gamma_0^a = 0$, because by definition $\delta\Gamma/\delta\phi = -J$, and the subscript 0 denotes evaluation at $\phi = \langle\phi\rangle$, corresponding precisely to $J = 0$.

We can read off the relation between derivatives of W and of Γ by comparing eqs. (21) and (22):

$$\Gamma_0^{ab} = -W_0^{-1ab} \quad , \quad \Gamma_0^{acb} = W_0^{-1aa'}W_0^{-1bb'}W_0^{-1cc'}W_0^{a'b'c'} \tag{24}$$

Note that I have accounted for the possibility that ϕ^b and ϕ^c are Grassmann, in which case flipping the order of $\phi^c - \langle\phi^c\rangle$ and $\phi^b - \langle\phi^b\rangle$ is comenspated in sign by flipping the order of b, c in Γ. The W's are assumed to be bosonic i.e. if the two-point function involves fermions, it must involve pairs of fermions; and the three-point function is assumed to be overall bosonic as well. The second expression can be rewritten as

$$\langle\phi^{a'}\phi^{b'}\phi^{c'}\rangle_c = -W_0^{a'b'c'} = \frac{1}{i}W_0^{a'a}\frac{1}{i}W_0^{b'b}\, i\Gamma_0^{abc}\,\frac{1}{i}W_0^{cc'}\,, \tag{25}$$

where I have compensated for the sign change associated with flipping the order of c, c' in W by flipping the order of b, c in Γ (in the event that b, c – and thus b', c' – are Grassmann). This matches precisely eq. (5) if one interprets $\phi^{a'} \to A^\nu(x_0)$, $\phi^{b'} \to \Psi_\alpha(x_1)$, $\phi^{c'} \to \bar\Psi_\beta(x_2)$. Thus, Γ_0^{abc} can be interpreted as the cubic vertex V of eq. (5).

Two additional side comments. (1) With the above choices, $\Gamma_0^{abc} \to \Gamma_0^{A\bar\Psi\Psi}$ (because $W_0^{b'b} \to \langle\Psi\bar\Psi\rangle$). The order makes complete sense, because you are supposed to read Γ like an action i.e. $\Gamma = ... + A_\mu\bar\Psi_\rho\Psi_\sigma\Gamma_0^{A_\mu\bar\Psi_\rho\Psi_\sigma} + ... = ... + \bar\Psi_\rho A_\mu\Gamma_0^{A_\mu\bar\Psi_\rho\Psi_\sigma}\Psi_\sigma + ...$ (2) Since $W_0^{b'b}$ is the second derivative of W, it must be symmetric if b' and b are bosonic, or antisymmetric if b' and b are Grassmann. In other words, we are saying $\langle\Psi_\alpha(x_1)\bar\Psi_\rho(y_1)\rangle = -\langle\bar\Psi_\rho(y_1)\Psi_\alpha(x_1)\rangle$. Note however in general: $\langle\Psi_\alpha(x_1)\bar\Psi_\rho(y_1)\rangle \neq \langle\Psi_\rho(y_1)\bar\Psi_\alpha(x_1)\rangle$, as one can see explicitly by checking out the case of the free fermion propagator.

Application to U(1) gauge invariance. We are now ready to use this formalism. Consider the path integral:

$$e^{iW(J)} = \int d\phi\, e^{i[S_{\rm GI}(\phi)+S_{\rm gf}(\phi)]+i\int d^4x\phi J} \tag{26}$$

where ϕ abstractly denotes all fields (photons and fermions) and J their respective sources. We divide the classical action S into its gauge invariant part and the gauge-fixing part. Suppose $S_{\rm GI}$ (and the measure) is invariant under $\phi \to \phi + \delta\phi$. We see that

$$\delta W = \langle\delta S_{\rm gf}\rangle_J + \int d^4x \langle\delta\phi\rangle_J J \tag{27}$$

where the subscipt J is supposed to remind us that these expectation values are evaluated with some non-zero J. Our symmetry of interest: $\delta\Psi = -ie\lambda\Psi$, $\delta\bar\Psi = ie\lambda\bar\Psi$, $\delta A_\mu = -\partial_\mu\lambda$ is such that the variation abstractly denoted by $\delta\phi$ is either independent of fields, or at most linear in the fields. Thus, $\langle\delta\phi\rangle_J = \delta\langle\phi\rangle_J$, where $\langle\phi\rangle_J = \delta W/\delta J$ is precisely what we mean by the field ϕ in the context of the Legendre transform to obtain $\Gamma(\phi)$. Similarly, our $\delta S_{\rm gf}$ is linear in the field (A^μ), and so $\langle\delta S_{\rm gf}\rangle_J = \delta S_{\rm gf}(\langle\phi\rangle_J)$. Thus, we infer that

$$\delta\Gamma = \delta W - \int d^4x\, \delta\phi\, J = \delta S_{\rm gf}(\phi)\,, \tag{28}$$

where ϕ should be understood as $\langle\phi\rangle_J = \delta W/\delta J$. Thus, we come to the conclusion that Γ can be written as

$$\Gamma(\phi) = \hat\Gamma(\phi) + S_{\rm gf}(\phi) \tag{29}$$

with the understanding that $\hat\Gamma(\phi)$ is gauge invariant i.e. $\delta\hat\Gamma = 0$; the only non-invariant part of $\Gamma(\phi)$ comes from the gauge-fixing term $S_{\rm gf}(\phi)$. Since the quantum action Γ is supposed to include all loop effects, this is telling us that loops do not generate extra non-invariant terms beyond what already exists in S_{gf}. Thus, we are led to $Z_1 = Z_2$.

This proof might seem a bit too slick to you. As a check, let us deduce the Ward-Takahashi identity using this formalism. The quantum action Γ has many terms, two of which are:

$$\Gamma = ... + \int d^4x\, d^4y\, d^4z\, A_\mu(x)\, \bar{\Psi}_\rho(y)\, \Gamma_0^{A_\mu \bar{\Psi}_\rho \Psi_\sigma}(x,y,z)\, \Psi_\sigma(z) + \int d^4y\, d^4z\, \bar{\Psi}_\rho(y) \Gamma_0^{\bar{\Psi}_\rho \Psi_\sigma}(y,z) \Psi_\sigma(z) + ... \tag{30}$$

Neither of these two terms coincide with the gauge-fixing term, and thus they must fall under $\hat{\Gamma}$ which is invariant. Let's work out their variation under the gauged U(1), isolate the terms that go like $\bar{\Psi}\Psi$ (these will be the only such terms in $\delta\hat{\Gamma}$), and set them to zero:

$$0 = \int d^4x\, d^4y\, d^4z\, (-\partial_\mu \lambda(x)) \bar{\Psi}_\rho(y)\, \Gamma_0^{A_\mu \bar{\Psi}_\rho \Psi_\sigma}(x,y,z)\, \Psi_\sigma(z) + \int d^4y\, d^4z\, \bar{\Psi}_\rho(y) \Gamma_0^{\bar{\Psi}_\rho \Psi_\sigma}(y,z) \Psi_\sigma(z) [ie\lambda(y) - ie\lambda(z)] \tag{31}$$

Since the field can be arbitrarily chosen by tuning the external sources J, we must have:

$$\partial_\mu \Gamma_0^{A_\mu \bar{\Psi}_\rho \Psi_\sigma}(x,y,z) = i\, e\, \Gamma_0^{\bar{\Psi}_\rho \Psi_\sigma}(y,z)[\delta(x-z) - \delta(x-y)] \tag{32}$$

Recall we have shown $\Gamma_0^{A_\mu \bar{\Psi}_\rho \Psi_\sigma}(x,y,z)$ is precisely the cubic vertex $V^\mu{}_{\rho\sigma}(x,y,z)$, and $\Gamma_0^{\bar{\Psi}_\rho \Psi_\sigma}(y,z) = -W_0^{-1\,\bar{\Psi}_\rho \Psi_\sigma}(y,z) = -S^{-1}{}_{\rho\sigma}(y,z)$ (eq. 24). Thus, we have reproduced exactly eq. (10) which is the Ward-Takahashi identity in real space.

Comment: it's interesting that the derivation using Ward identity (conservation of current J_N) makes use of the *global* U(1), while the derivation from the quantum action uses the *local* U(1).