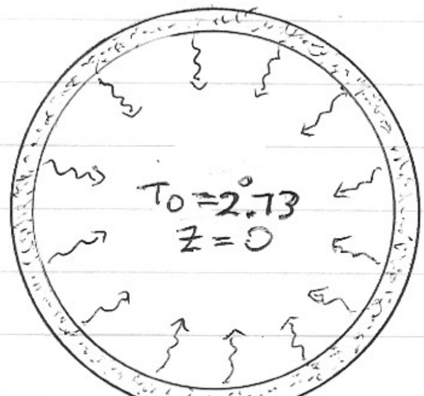
Cosmic Microwave Background

Blackbody radiation spectrum peaks at frequency ν_{max}



$z = 1100$
$T = 3000° K$

$$h\nu_{max} = 2.82\, kT$$

for $T_0 = 2.725$ K

$$\nu_{max} = 1.6 \times 10^{11} \text{ Hz}$$

$$c/\nu_{max} = 1.9 \text{ mm}$$

We will show that $T = T_0(1+z)$.
We already know that $\nu = \nu_0 (1+z)$.

Properties of blackbody radiation

specific energy density $\rightarrow$ $u_\nu(T) = \dfrac{8\pi h \nu^3}{c^3} \dfrac{1}{e^{h\nu/kT} - 1} \quad \left[\dfrac{J}{m^3\, Hz}\right]$

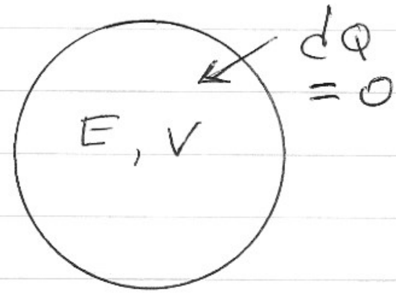
total energy density $\rightarrow$ $u(T) = \displaystyle\int_0^\infty u_\nu \, d\nu = \dfrac{4\sigma}{c} T^4 \quad \left[\dfrac{J}{m^3}\right]$

Make substitutions $\nu \rightarrow \nu_0(1+z)$
$T \rightarrow T_0(1+z)$

$$u(T) = \int_0^\infty (1+z)^4 \frac{8\pi h \nu_0^3}{c^3} \frac{1}{e^{h\nu_0/kT_0} - 1} \, d\nu_0 = (1+z)^4 \frac{4\sigma T_0^4}{c}$$

$$= \frac{4\sigma}{c} T^4$$

The energy in radiation follows the first law of thermodynamics



heat

$$dQ = dE + P\,dV$$

$$\frac{dE}{dt} = -P\frac{dV}{dt}$$

$$E = uV = \frac{4\sigma T^4}{c}V$$

$$P = \frac{u}{3} = \frac{4\sigma T^4}{3c} \qquad \text{radiation pressure}$$

$$\frac{d}{dt}\left(T^4 V\right) = -\frac{T^4}{3}\frac{dV}{dt}$$

$$4T^3\frac{dT}{dt}V + T^4\frac{dV}{dt} = -\frac{T^4}{3}\frac{dV}{dt}$$

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{3V}\frac{dV}{dt}$$

Volume is proportional to a^3

$$dV = 3a^2\,da$$

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{a}\frac{da}{dt}$$

$$\frac{d}{dt}(\ln T) = -\frac{d}{dt}(\ln a)$$

$$\int_T^{T_0} d(\ln T) = -\int_a^1 d(\ln a)$$

$$\ln T_0 - \ln T = \ln a$$

$$\ln(T/T_0) = -\ln a$$

$$T = T_0 a^{-1}$$

$$T = T_0 (1+z)$$

Note that energy density of radiation scales as $T^4 \propto (1+z)^4$ while the energy density of matter scales as $(1+z)^3$.

At present $\rho_m c^2 \gg u_r$, but in the distant past, the opposite was true.

$\rho_m c^2$ = energy density of matter $= u_m$

$u_r = \dfrac{4\sigma T^4}{c}$ = energy density of radiation

You can also write $u_r = \rho_r c^2$