The Consensus Model

The general form of the Friedman Equation is

\[
\left( \frac{1}{a} \frac{da}{dt} \right)^2 = \frac{8\pi G}{3} \left( \varrho_r + \varrho_m + \varrho_{\Lambda} \right) \]

[Equation 23]

Observations which we will discuss have led to a consensus model in which the Universe is "flat", which means \( k = 0 \). This means that the varying densities of radiation and matter, and the constant density of \( \Lambda \), must add to make \( \Omega = 1 \) at all times.

In terms of their present densities:

\[
\varrho_r = \frac{\varrho_{\text{r,0}}}{a^4}, \quad \varrho_m = \frac{\varrho_{\text{m,0}}}{a^3}, \quad \varrho_{\Lambda} = \varrho_{\Lambda,0}
\]

Their present contributions to \( \Omega \) are

\[
\Omega_{\text{r,0}} = \frac{8\pi G \varrho_{\text{r,0}}}{3H_0^2}, \quad \Omega_{\text{m,0}} = \frac{8\pi G \varrho_{\text{m,0}}}{3H_0^2}
\]

\[
\Omega_{\Lambda,0} = \frac{8\pi G \rho_{\Lambda,0}}{3H_0^2}, \quad \text{where} \quad \frac{3H_0^2}{8\pi G} = \rho_c, \quad \text{the critical density}
\]

Use these relations to replace \( \varrho \) with \( \Omega \).
With $k=0$, Equation (23) becomes

\[
\frac{da}{dt} = H_0 \left( \frac{\Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} a^2}{a} \right)^{1/2}
\]

At different times in the history of the Universe, one of radiation, matter, or $\Lambda$ dominates the mass/energy density. For example, at the earliest times, radiation dominated, and therefore

\[
\frac{dg}{dt} = H_0 \Omega_{r,0}^{1/2} \quad \text{when } z \gg 3300
\]

\[
\int a \, da = H_0 \Omega_{r,0}^{1/2} \int_0^t dt
\]

\[
a(t) = 2 H_0 \Omega_{r,0}^{1/2} t
\]

\[
a(t) = \left[ 2 H_0 \Omega_{r,0}^{1/2} \right]^{1/2} t^{1/2}
\]

At later times, matter dominated, and

\[
\frac{da}{dt} = H_0 \left( \frac{\Omega_{m,0}}{a^2} \right)^{1/2}, \quad \text{whose solution is}
\]

\[
a(t) = \left[ \frac{3}{2} \Omega_{m,0} H_0 \right]^{2/3} t^{2/3}
\]

which we showed earlier in "Newtonian" cosmology.
In the future, when $a \gg 0.6$, $\Lambda$ will dominate, and

$$\frac{da}{dt} = H_0 \Omega_{\Lambda,0}^{1/2} a,$$

with solution

$$ln a = H_0 \Omega_{\Lambda,0}^{1/2} t$$

$$a = e^{H_0 \Omega_{\Lambda,0}^{1/2} t}$$

and the Universe will expand exponentially with time constant $T$, where

$$T = \frac{1}{H_0 \Omega_{\Lambda,0}^{1/2}}$$

We will see that $\Omega_{\Lambda,0} \approx 0.7$, therefore $T \approx 16.5 \times 10^9$ yr. In this phase, the acceleration is positive, unlike the radiation and matter dominated eras.

You can see explicitly the contributions to acceleration by starting with Eq. (25) squared

$$\left(\frac{da}{dt}\right)^2 = H_0^2 \left(\frac{\Omega_{\Lambda,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{k,0} a^2\right)$$

and differentiate...
When did the Universe switch from negative acceleration to positive acceleration?

When \( \frac{\Omega m_{\text{tot}}}{2a^2} \)

\[
(1+z) = \frac{1}{a} = \left( \frac{2\Omega m_{\text{tot}}}{\Omega m_{\text{tot}}} \right)^{1/3} = \left( \frac{2 \times 0.7}{0.3} \right)^{1/3}
\]

\( z = 0.67 \)

The present age of the Universe is very close to \( H^{-1} \), by coincidence, because it has been accelerating for approximately the same amount of time that it was decelerating.

\( t_0 \approx 0.993 \frac{1}{H_0} = 13.7 \times 10^9 \text{ yr} \)