The Consensus Model

The general form of the Friedman Equation is

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}\left(\rho_r + \rho_m + \rho_\Lambda\right)\right] a^2 = -kc^2 \quad (23)$$

Observations which we will discuss have led to a consensus model in which the Universe is "flat", which means $k=0$. This means that the varying densities of radiation and matter, and the constant density of Λ, must add to make $\Omega = 1$ at all times.

In terms of their present densities:

$$\rho_r = \frac{\rho_{r,0}}{a^4}, \quad \rho_m = \frac{\rho_{m,0}}{a^3}, \quad \rho_\Lambda = \rho_{\Lambda,0}$$

Their <u>present</u> contributions to Ω are

$$\Omega_{r,0} = \frac{8\pi G}{3H_0^2}\rho_{r,0}, \quad \Omega_{m,0} = \frac{8\pi G}{3H_0^2}\rho_{m,0}$$

$$\Omega_{\Lambda,0} = \frac{8\pi G}{3H_0^2}\rho_{\Lambda,0}, \quad \text{where } \frac{3H_0^2}{8\pi G} = \rho_{c,0}$$

the critical density

Use these relations to replace ρ with Ω

With $k=0$, Equation (23) becomes

(25) $$\boxed{\frac{da}{dt} = H_0 \left(\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}\, a^2 \right)^{1/2}}$$

At different times in the history of the Universe, one of radiation, matter, or Λ dominates the mass/energy density.
For example, at the earliest times radiation dominated, and therefore

$$\frac{da}{dt} = \frac{H_0 \Omega_{r,0}^{1/2}}{a} \qquad \text{when } z \ggg 3300$$

$$\int_0^a a\, da = H_0 \Omega_{r,0}^{1/2} \int_0^t dt$$

$$a^2(t) = 2 H_0 \Omega_{r,0}^{1/2}\, t$$

$$a(t) = \left[2 H_0 \Omega_{r,0}^{1/2} \right]^{1/2} t^{1/2}$$

At later times matter dominated, and

$$\frac{da}{dt} = H_0 \left(\frac{\Omega_{m,0}}{a} \right)^{1/2}$$, whose solution is

$$a(t) = \left[\frac{3}{2} \Omega_{m,0}^{1/2} H_0 \right]^{2/3} t^{2/3}$$

which we showed earlier in "Newtonian" cosmology.

In the future, when $a >> 0.6$, Λ will dominate, and

$$\frac{da}{dt} = H_0 \Omega_{\Lambda,0}^{1/2} \, a \quad , \qquad \text{with solution}$$

$$\ln a = H_0 \Omega_{\Lambda,0}^{1/2} \, t$$

$$a = e^{H_0 \Omega_{\Lambda,0}^{1/2} t}$$

and the Universe will expand exponentially with time constant τ, where

$$\tau = \frac{1}{H_0 \Omega_{\Lambda,0}^{1/2}}$$

We will see that $\Omega_{\Lambda,0} \simeq 0.7$, therefore $\tau \simeq 16.5 \times 10^9$ yr. In this phase, the acceleration is positive, unlike the radiation and matter dominated eras.

You can see explicitly the contributions to acceleration by starting with Eq. (25) squared

$$\left(\frac{da}{dt}\right)^2 = H_0^2 \left(\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} \, a^2 \right)$$

and differentiate ...

$$\frac{d}{dt}\left(\frac{da}{dt}\right)^2 = H_0^2\left(-\frac{2\Omega_{r,0}}{a^3} - \frac{\Omega_{m,0}}{a^2} + 2\Omega_{\Lambda,0}\, a\right)\frac{da}{dt}$$

$$2\frac{da}{dt}\frac{d^2a}{dt^2} = H_0^2\left(-\frac{2\Omega_{r,0}}{a^3} - \frac{\Omega_{m,0}}{a^2} + 2\Omega_{\Lambda,0}\, a\right)\frac{da}{dt}$$

(26) $$\boxed{\frac{d^2a}{dt^2} = H_0^2\left(-\frac{\Omega_{r,0}}{a^3} - \frac{\Omega_{m,0}}{2a^2} + \Omega_{\Lambda,0}\, a\right)}$$

When did the Universe switch from negative acceleration to positive acceleration?

When $\dfrac{\Omega_{m,0}}{2a^2} = \Omega_{\Lambda,0}\, a$

$$(1+z) = \frac{1}{a} = \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3} \approx \left(\frac{2\times 0.7}{0.3}\right)^{1/3}$$

$$z = 0.67$$

The present age of the Universe is very close to H_0^{-1}, by coincidence, because it has been accelerating for approximately the same amount of time that it was decelerating.

$$t_0 \approx \frac{0.993}{H_0} = 13.7\times 10^9\ \text{yr}$$