Relativistic Cosmology (cont.)

\[
\left( \frac{1}{a} \frac{da}{dt} \right)^2 - \frac{8\pi G}{3} \rho - \frac{1}{3} \Lambda c^2 \right] a^2 = -hc^2 \quad (21)
\]

\[
\frac{d^2 a}{dt^2} = \left[ \frac{-4\pi G}{3} \left( \rho + \frac{3\rho_r}{c^2} \right) + \frac{1}{3} \Lambda c^2 \right] a \quad (22)
\]

where \( \rho = \rho_r + \rho_m \)

The cosmological constant \( \Lambda \) acts as a constant mass density \( \rho_\Lambda \)

\[ \rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \]

So the Friedmann equation (21) can be written

\[
\left( \frac{1}{a} \frac{da}{dt} \right)^2 - \frac{8\pi G}{3} \left( \rho + \frac{3\rho_r}{c^2} + \rho_\Lambda \right) \right] a^2 = -hc^2 \quad (23)
\]

\( \Lambda \) also acts as a negative pressure \( P_\Lambda \)

\[ P_\Lambda = -\rho_\Lambda c^2 \]

So \[ \frac{1}{3} \Lambda c^2 = \frac{8\pi G}{3} \rho_\Lambda = -\frac{8\pi G}{3} \frac{P_\Lambda}{c^2} \]
Therefore, we can also write

\[ \frac{1}{3} \Lambda c^2 = -\frac{4\pi G}{3} \rho_r - \frac{4\pi G}{c^2} P_\Lambda \]

The acceleration equation (22) can be written in a more symmetric way:

\[ \frac{d^2 g}{dt^2} = -\frac{4\pi G}{3} \left[ \rho_r + \rho_m + \rho_r + 3 \left( \frac{P_r + P_\Lambda}{c^2} \right) \right] a \tag{24} \]

Since \( \Lambda \) is a constant energy density, the total amount of dark energy grows with the scale factor \( a \), but it doesn't violate the 1st law of thermodynamics:

\[ \frac{dS}{dt} = -3 \left( \rho + \frac{P}{c^2} \right) \frac{1}{a} \frac{dg}{dt} \tag{16} \]

because it has negative pressure \( (P_\Lambda = -\rho_\Lambda c^2) \). Therefore, \( \Lambda \) contributes a positive term to the acceleration equation (24), which dominates if and when the scale factor \( a \) becomes large enough.
Hubble Diagram (in a matter-only universe)

\[ \begin{align*}
\text{m} - \text{M} & = 5 \log d_L - 5 \\
\text{for } \Omega_m = 0 & \quad d_L = \frac{cz}{H_0} \left[ \frac{1+z}{2} \right] \\
\Omega_m = 1 & \quad d_L = 2c \left[ 1+z - \sqrt{1+z} \right] \frac{1}{H_0} \\
\Omega_m = 2 & \quad d_L = \frac{cz}{H_0} \\
\text{When } z > 0.1 & \quad d_L = \frac{cz}{H_0} \text{ for all } \Omega
\end{align*} \]
What is luminosity distance \( d_L \)?

\[
\text{Flux } F = \frac{L}{4\pi d_L^2} = \frac{L}{4\pi (1+z)^2 r^2} 
\]

\( r(t) = a(t) \, r_0 \)

\( r = r_0 \) at the present time

coordinate distance

So \( d_L = (1+z) \, r_0 \)

One factor of \( 1+z \) comes from the redshift of the light, and another factor of \( 1+z \) comes from cosmological time dilation, the fact that photons arrive at a lower rate. So the energy flux is reduced by \( \frac{1}{1+(1+z)^2} \).

Coordinate distance is difficult to calculate, except for the case \( \Sigma_m = 1 \), where a photon travels a distance \( dr = c \, dt \) in a time \( dt \), where \( dr = a(t) \, dr_0 \). Therefore

\[
\int^{r}_{r_0} dr_0 = \int^{t}_{0} \frac{c \, dt}{a(t)} 
\]

Previously we showed that \( a(t) = \left( \frac{t}{t_0} \right)^{2/3} \) for \( \Sigma_m = 1 \).
\[ r_0 = t_0^{2/3} \int_0^{t_0} \frac{c \, d \tau}{\tau^{2/3}} = 3 \, c \, t_0^{2/3} \left( t_0^{1/3} - t^{1/3} \right) \]

At the present time \( r_0 = r_1 \), so

\[ r = 3 \, c \, t_0 \left[ 1 - \left( \frac{t}{t_0} \right)^{1/3} \right] \]

\[ = 3 \, c \, t_0 \left[ 1 - \sqrt[3]{a(t)} \right] \quad a(t) = \left( \frac{t}{t_0} \right)^{2/3} \]

But \( a(t) = \frac{1}{1+z} \) so

\[ r = 3 \, c \, t_0 \left[ 1 - \frac{1}{\sqrt[3]{1+z}} \right] \]

Previously we showed that

\[ t = \frac{2 \, z}{3 \, H_0} \left[ \frac{1}{1+z} \right]^{3/2} \quad \text{for } \Omega_m = 1 \]

\[ \Rightarrow t_0 = \frac{2 \, z}{3 \, H_0} \quad \text{is the present age} \]

\[ \Rightarrow r = \frac{2 \, c}{H_0} \left[ 1 - \frac{1}{\sqrt[3]{1+z}} \right] \]

\( \frac{2c}{H_0} \) is the "horizon distance", the furthest observable point \( (z = \infty) \)
Finally, the luminosity distance is

\[ d_L = (1+z) \frac{r}{H_0} = \frac{2c}{H_0} \left[ 1 + z - \sqrt{1+z} \right] \]

Aside: It is interesting that a photon travels a distance \( r_0 = \frac{2c}{H_0} \) from the horizon in a time \( t_0 = \frac{2}{3H_0} \)

\[ \Rightarrow r_0 = 3c \cdot t_0 \]

The photon travels farther than \( c \cdot t_0 \) because the Universe was expanding while it travelled.

Note: It is difficult to calculate \( d_L \) for other values of \( \Omega \) because it is not the case that \( dr = a(t) \, dr_0 \) for other \( \Omega \)'s, \( \Omega = 1 \) \( (k = 0) \) is the special case in which space is "flat". Other cases are "curved" space, where \( r \) can be either greater than or less than \( r_0 \) at \( t_0 \).
Angular Diameter - Redshift Relation

Consider a galaxy of physical diameter $D$ at comoving coordinate $r_0$. In the small-angle limit:

$$D = a(t) \frac{r_0 \theta}{D}$$

$$D = \left(\frac{1}{1+z}\right) r_0 \theta$$

$$\theta = \frac{(1+z)}{r_0} D$$

but $d_L = (1+z) r_0$

$$\Rightarrow \quad \theta = \frac{(1+z)^2}{d_L} D$$

In Euclidean space $\theta = \frac{D}{d}$

\[
\frac{\theta}{D} \left[ \frac{\text{rad}}{\text{kpc}} \right]
\]

\[
\frac{1}{8.15}
\]
Surface brightness is a function of redshift, unlike in Euclidean space. Surface brightness is flux per unit solid angle:

\[ SB = \frac{L}{4\pi d_L^2 / \pi (\theta/2)^2} \]

But \( \theta = \frac{(1+z)^2 D}{d_L} \)

\[ \Rightarrow SB = \frac{L}{(\pi D)^2 (1+z)^4} \]

In Euclidean space, \( SB = \frac{L}{(\pi D)^2} \)

Cosmological surface brightness dimming is proportional to \( \frac{1}{(1+z)^4} \)