Relativistic Cosmology (cont.)

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi G\rho}{3} - \frac{1}{3}\Lambda c^2\right]a^2 = -kc^2 \quad (21)$$

$$\frac{d^2a}{dt^2} = \left[-\frac{4\pi G}{3}\left(\rho + \frac{3P_r}{c^2}\right) + \frac{1}{3}\Lambda c^2\right]a \quad (22)$$

where $\rho = \rho_r + \rho_m$

The cosmological constant Λ acts as a <u>constant</u> mass density ρ_Λ

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}$$

So the Friedmann equation (21) can be written

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}(\rho_r + \rho_m + \rho_\Lambda)\right]a^2 = -kc^2 \quad (23)$$

Λ also acts as a negative pressure P_Λ

$$P_\Lambda = -\rho_\Lambda c^2$$

So $$\frac{1}{3}\Lambda c^2 = \frac{8\pi G}{3}\rho_\Lambda = -\frac{8\pi G}{3}\frac{P_\Lambda}{c^2}$$

Therefore, we can also write

$$\frac{1}{3}\Lambda c^2 = -\frac{4\pi G}{3}\rho_\Lambda - \frac{4\pi G\, P_\Lambda}{c^2}$$

The acceleration equation (22) can be written in a more symmetric way:

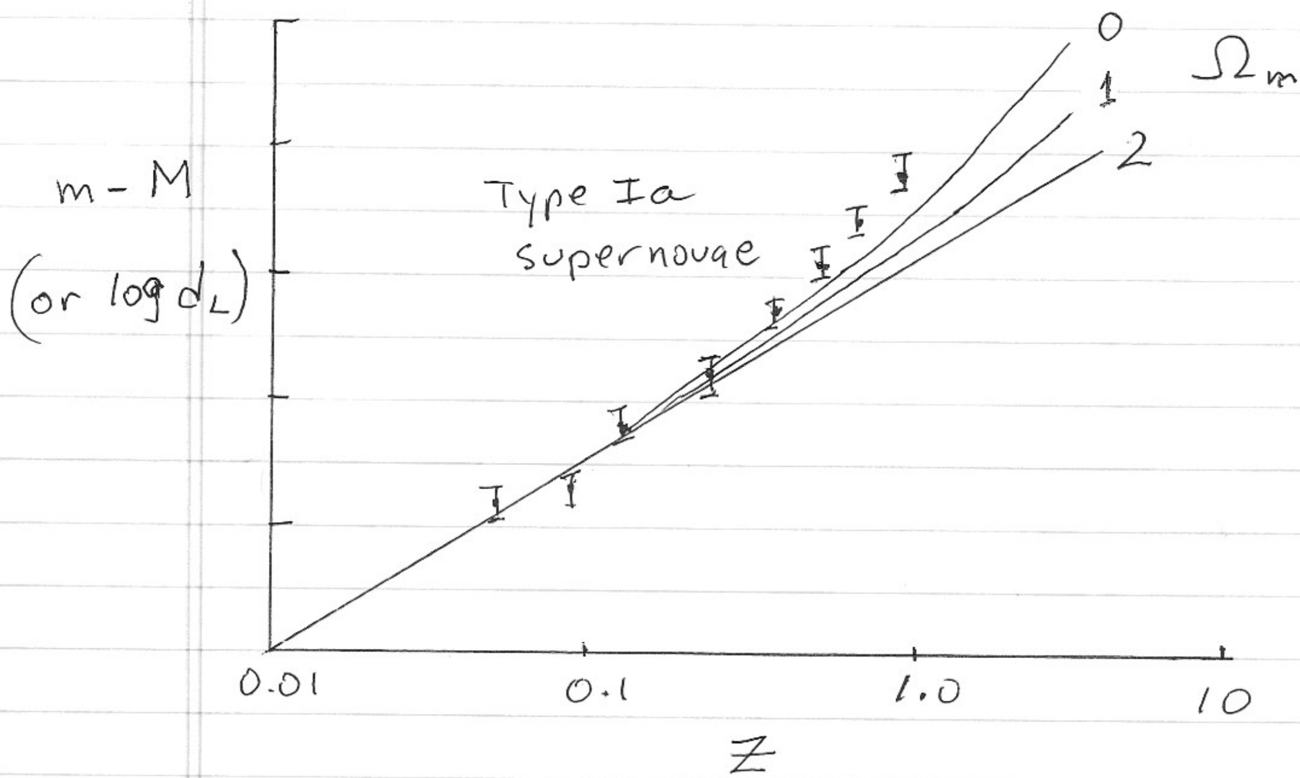
$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left[\rho_r + \rho_m + \rho_\Lambda + 3\left(\frac{P_r + P_\Lambda}{c^2}\right)\right] a \qquad (24)$$

Since Λ is a constant energy density the total amount of dark energy grows with the scale factor a, but it doesn't violate the 1st law of thermodynamics

$$\frac{d\rho}{dt} = -3\left(\rho + \frac{P}{c^2}\right)\frac{1}{a}\frac{da}{dt} \qquad (16)$$

because it has <u>negative</u> pressure ($P_\Lambda = -\rho_\Lambda c^2$). Therefore, Λ contributes a positive term to the acceleration equation (24), which dominates if and when the scale factor a becomes large enough.

Hubble Diagram
(in a matter-only universe)



$$m - M = 5 \log d_L - 5$$

for $\Omega_m = 0$ $\qquad d_L = \frac{cz}{H_0}\left[1 + \frac{z}{2}\right]$

$\Omega_m = 1$ $\qquad d_L = \frac{2c}{H_0}\left[1 + z - \sqrt{1+z}\right]$

$\Omega_m = 2$ $\qquad d_L = \frac{cz}{H_0}$

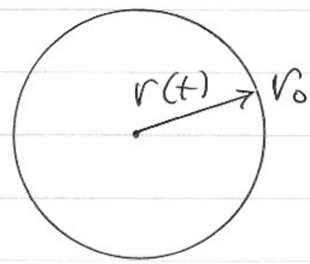
When $z < 0.1$ $\quad d_L \simeq \frac{cz}{H_0}$ for all Ω

What is luminosity distance d_L ?

Flux $F = \dfrac{L}{4\pi d_L^2} = \dfrac{L}{4\pi (1+z)^2 r^2}$

$r(t) = a(t)\, r_0$ $\quad$ $r = r_0$ at the present time
↑ coordinate distance

So $d_L = (1+z)\, r_0$



One factor of $1+z$ comes from the redshift of the light, and another factor of $1+z$ comes from cosmological time dilation, the fact that photons arrive at a lower rate.
So the energy flux is reduced by $1/(1+z)^2$.

Coordinate distance is difficult to calculate, except for the case $\Omega_m = 1$, where a photon travels a distance $dr = c\,dt$ in a time dt, where $dr = a(t)\,dr_0$. Therefore

$$\int_{r_0}^{0} dr_0 = \int_{t}^{t_0} \frac{c\,dt}{a(t)}$$

Previously we showed that $a(t) = (t/t_0)^{2/3}$ for $\Omega_m = 1$.

$$r_0 = t_0^{2/3} \int_t^{t_0} \frac{c\,dt}{t^{2/3}} = 3c t_0^{2/3} (t_0^{1/3} - t^{1/3})$$

At the present time $r_0 = r$, so

$$r = 3ct_0 \left[1 - \left(\frac{t}{t_0}\right)^{1/3} \right]$$

$$= 3ct_0 \left[1 - \sqrt{a(t)} \right] \qquad a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

But $a(t) = \frac{1}{1+z}$ so

$$r = 3ct_0 \left[1 - \frac{1}{\sqrt{1+z}} \right]$$

Previously we showed that

$$t = \frac{2}{3H_0} \left[\frac{1}{1+z} \right]^{3/2} \qquad \text{for } \Omega_m = 1$$

$\Rightarrow$ $t_0 = \frac{2}{3 \cdot H_0}$ is the present age

$$\Rightarrow r = \frac{2c}{H_0} \left[1 - \frac{1}{\sqrt{1+z}} \right]$$

$\frac{2c}{H_0}$ is the "horizon distance", the furthest observable point ($z = \infty$)

Finally, the luminosity distance is

$$d_L = (1+z)\, r = \frac{2c}{H_0}\left[1+z-\sqrt{1+z}\right]$$

Aside: It is interesting that a photon travels a distance $r_0 = \frac{2c}{H_0}$ from the horizon

in a time $t_0 = \frac{2}{3H_0}$

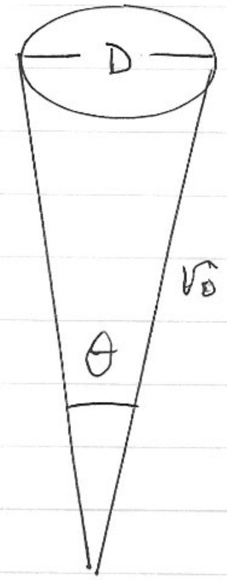
$\Rightarrow \quad r_0 = 3ct_0$

The photon travels farther than ct_0 because the Universe was expanding while it travelled.

Note: It is difficult to calculate d_L for other values of Ω because it is not the case that $dr = a(t)\, dr_0$ for other Ω's, $\Omega = 1$ $(k=0)$ is the special case in which space is "flat". Other cases are "curved" space, where r can be either greater than or less than r_0 at t_0.

Angular Diameter - Redshift Relation

Consider a galaxy of physical diameter D at comoving coordinate r_0. In the small-angle limit



$$D = a(t)\, r_0\, \theta$$

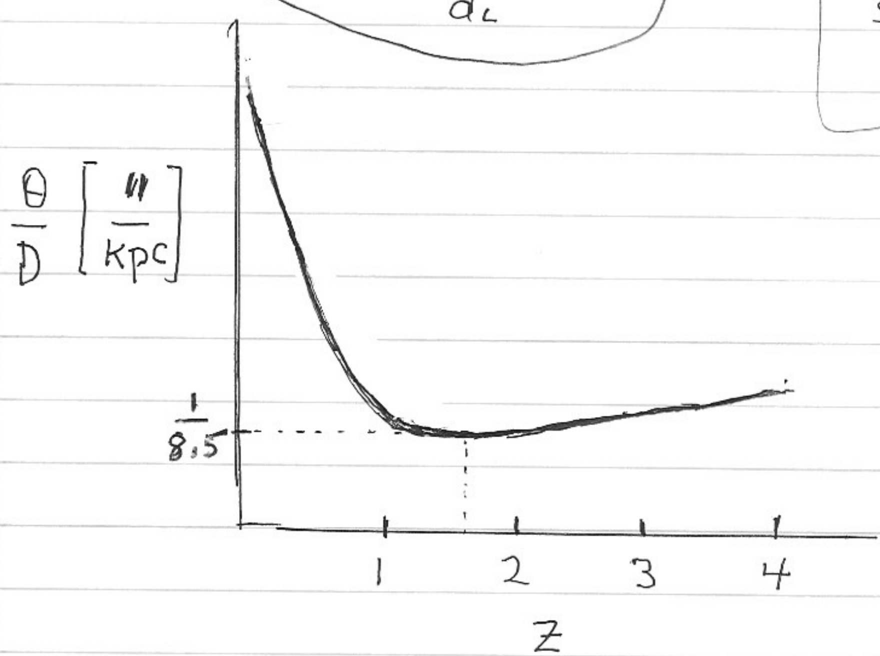
$$D = \left(\frac{1}{1+z}\right) r_0\, \theta$$

$$\theta = \frac{(1+z)\, D}{r_0}$$

but $d_L = (1+z)\, r_0$

$$\Rightarrow \quad \theta = \frac{(1+z)^2\, D}{d_L}$$

In Euclidean space $\theta = \dfrac{D}{d}$



Surface brightness is a function of redshift, unlike in Euclidean space. Surface brightness is flux per unit solid angle

$$SB = \frac{L}{4\pi d_L^2} \Big/ \pi(\theta/2)^2$$

but $\theta = \dfrac{(1+z)^2 D}{d_L}$

$$\Rightarrow \boxed{SB = \frac{L}{(\pi D)^2} \frac{1}{(1+z)^4}}$$

In Euclidean space, $SB = \dfrac{L}{(\pi D)^2}$

Cosmological surface brightness dimming is proportional to $\dfrac{1}{(1+z)^4}$