"Newtonian" Cosmology (cont.)

Last time we showed that for an $\Omega = 1$ matter-dominated universe, the scale factor a increases as

$$a = (t/t_0)^{2/3}$$

and the age of the universe at redshift z is

$$t = \frac{2}{3H_0} \frac{1}{(1+z)^{3/2}}$$

The present age of the universe (at $z=0$) would be

$$t_0 = \frac{2}{3H_0} = 9.2 \times 10^9 \text{ yr}$$

This is not compatible with the ages of the oldest stars in globular clusters $\simeq 13 \times 10^9$ yr. There appears to be no reason why Ω should equal 1 exactly. Adding up all of the matter in nearby clusters and voids has led to

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}} = 0.27 \pm 0.04$$

of which the baryons contribute $\Omega_{b,0} = 0.044 \pm 0.004$

The rest of Ω_m is from dark matter.

Even $\Omega_0 = 0.27$ seems unlikely, because it would require that in the past, with

$$\Omega = 1 + \left(\frac{\Omega_0 - 1}{1 + \Omega_0 z}\right)$$

from Equation (8) of "Accompanying notes"

Ω starts out almost exactly equal to 1 at high z. For example, at radiation-matter equality, when $z \simeq 3300$, $\Omega = 0.9992$. This coincidence continues to be a defining paradox of cosmology, that Ω is so close to 1.

The matter-dominated analysis is incomplete. In the next section we will generalize the dynamical equations of the universe to include radiation, and finally to include dark energy. But the coincidence that $\Omega = 1$ will remain.

Relativistic Cosmology

Both matter and radiation gravitate according to their equivalent mass and energy. Define $\rho = \rho_m + \rho_r$ and $u = u_m + u_r$, where

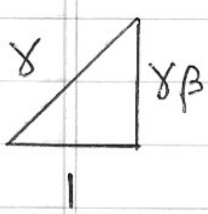
$$\rho_m c^2 = u_m$$
$$\rho_r c^2 = u_r = (4\sigma/c)\, T^4$$

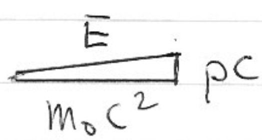
Here T refers to the radiation temperature since the temperature of matter contributes little energy compared with its rest-mass energy, as long as its thermal (random) motions are non-relativistic ($v \ll c$).

In general, $\boxed{E^2 = (m_o c^2)^2 + (pc)^2}$

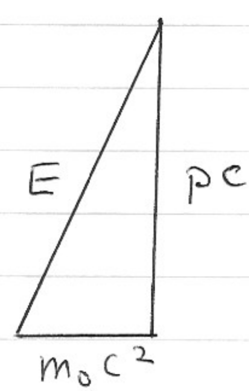
where momentum $p = \gamma m_o v = \dfrac{m_o v}{\sqrt{1-(v/c)^2}}$

For a non-relativistic particle, $\gamma \simeq 1$ and $pc \simeq m_o v c \ll m_o c^2$





non-relativistic



relativistic

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = v/c$$

The present density of matter, as measured by adding up all the mass in clusters and voids, is expressed as its ratio to the critical density

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}} = 0.27 \pm 0.04$$

Here $\rho_{c,0} = \frac{3 H_0^2}{8 \pi G} = 9.47 \times 10^{-27} \frac{kg}{m^3}$

The present density of radiation is

$$\rho_{r,0} = \frac{1}{c^2} \frac{4 \sigma T_0^4}{c} = 4.63 \times 10^{-31} \frac{kg}{m^3}$$

where $T_0 = 2.725$ K. Then the ratio

$$\frac{\rho_{m,0}}{\rho_{r,0}} = \frac{0.27 \times 9.47 \times 10^{-27}}{4.63 \times 10^{-31}} = 5500$$

Matter dominates now, but in the past

$$\rho_m = \rho_{m,0} (1+z)^3$$
$$\rho_r = \rho_{r,0} (1+z)^4 \qquad \text{since } T = T_0 (1+z)$$

where $\frac{1}{(1+z)} = a$, the scale factor

This means that matter and radiation density were equal when

$$\rho_{r,0}(1+z)^4 = \rho_{m,0}(1+z)^3$$

$$1+z = \frac{\rho_{m,0}}{\rho_{r,0}} = 5500$$

This was well before decoupling of matter and radiation at $z \simeq 1100$, so we can't directly observe the radiation dominated era.

To be more complete, any relativistic particles, defined as having $E \gg m_0 c^2$, behave like radiation in terms of their energy density. So we have to add neutrinos produced in the early universe by reactions such as

$$n \to p + e^- + \bar{\nu}$$
$$p \to n + e^+ + \nu$$

Neutrinos are weakly interacting, spin 1/2 particles with mass $\lesssim 0.1$ eV. They increase the effective radiation density by 68%, that is

$$\rho_{r,0} = 1.68\, \rho_{CMB,0}$$

This changes the epoch of matter and radiation equality to

$$1+z = \frac{\rho_{m,0}}{1.68\,\rho_{CMB,0}} \approx 3300$$

From now on ρ_r will refer to $(\rho_\nu + \rho_{CMB})$ and radiation pressure $P_r = (1/3)\rho_r c^2$. The pressure of matter is unimportant for the same reason that its thermal energy is much less than its rest energy.

We can now apply the first law of thermodynamics (energy conservation) to the total (matter + radiation) density,

$$\boxed{\rho = \rho_m + \rho_r}$$

$$dQ = dE + P\,dV = 0$$

$$\frac{dE}{dt} + P\frac{dV}{dt} = 0$$

$$\frac{d}{dt}(\rho c^2 a^3) + P_r \frac{d}{dt}(a^3) = 0 \qquad (1)$$

$$\frac{d\rho}{dt} c^2 a^3 + 3\rho c^2 a^2 \frac{da}{dt} + 3 P_r a^2 \frac{da}{dt} = 0$$

$$\boxed{\frac{d\rho}{dt} = -3\left(\rho + \frac{P_r}{c^2}\right)\frac{1}{a}\frac{da}{dt}}$$

Now we use the Friedman Equation

$$H^2a^2 - \frac{8\pi G a^2 \rho}{3} = -kc^2$$

(2) $$a\left(\frac{da}{dt}\right)^2 - \frac{8\pi G a^3 \rho}{3} = -kc^2 a$$

Take a time derivative to see how acceleration behaves

$$\left(\frac{da}{dt}\right)^3 + 2a\frac{da}{dt}\frac{d^2a}{dt^2} - \frac{8\pi G}{3}\frac{d}{dt}(\rho a^3) = -kc^2\frac{da}{dt}$$

Use Equation (1) on the previous page to replace the third term on the left-hand side

(3) $$\left(\frac{da}{dt}\right)^3 + 2a\frac{da}{dt}\frac{d^2a}{dt^2} + \frac{8\pi G}{3c^2}3P_r a^2\frac{da}{dt} = -kc^2\frac{da}{dt}$$

Combine Equations (2) and (3) to eliminate $\frac{da}{dt}$

$$\boxed{\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P_r}{c^2}\right)a}$$