Lorentz Transformations

\[ x' = \gamma (x - vt) \]
\[ y' = y \]
\[ z' = z \]
\[ t' = \gamma (t - \frac{vx}{c^2}) \]

Inverse Transformations (exchange prime and unprimed, change sign of \(v\))

\[ x = \gamma (x' + vt') \]
\[ y' = y' \]
\[ z = z' \]
\[ t = \gamma (t' + \frac{vx}{c^2}) \]

Time Dilation

\[ \Delta t = x \Delta t' \]

Transformation of Velocities

\[ \frac{dx'}{dt'} = \gamma \left( \frac{dx - vdt}{dt - \frac{vdx}{c^2}} \right) = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \]

\[ \frac{dy'}{dt'} = \gamma \left( \frac{dy}{dt - \frac{vdx}{c^2}} \right) = \frac{u_y}{\gamma (1 - \frac{vu_x}{c^2})} \]

\[ \frac{dz'}{dt'} = \frac{u_z}{\gamma (1 - \frac{vu_x}{c^2})} \]
Inverse Transformation of Velocities (exchange primed and unprimed, change sign of \(v\))

\[
\begin{align*}
    u_x &= \frac{u_x' + v}{1 + \frac{uv'}{c^2}} \\
    u_y &= \frac{u_y'}{\gamma(1 + \frac{uv'}{c^2})} \\
    u_z &= \frac{u_z'}{\gamma(1 + \frac{uv'}{c^2})}
\end{align*}
\]

Application of Transformation of Velocities to Aberration of Light

A ray of light has

\[
\begin{align*}
    u_x' &= c \cos \theta', \\
    u_y' &= c \sin \theta' \\
    u_x &= c \cos \theta, \\
    u_y &= c \sin \theta
\end{align*}
\]

\[
\begin{align*}
    c \cos \theta &= \frac{c \cos \theta' + v}{1 + \frac{uv'}{c^2}} \\
    c \sin \theta &= \frac{c \sin \theta'}{\gamma(1 + \frac{uv'}{c})}
\end{align*}
\]

This shows that \(\theta > \theta'\). In the case that \(\theta' = \pi/2\), it is useful to note that \(\sin \theta = 1/\gamma\). In the limit \(\gamma \gg 1\), \(\theta \approx 1/\gamma\). If the source is an isotropic emitter, half the radiation is beamed into an angle \(1/\gamma\).

\[
\begin{align*}
    S' &= \sin^{-1}(\frac{1}{\gamma}) \\
    S' &= \sin^{-1}(\frac{1}{\gamma})
\end{align*}
\]

Doppler effect - involves the arrival rate of wavefronts emitted by a moving source.

Let the frequency of radiation be \(v'\) in the rest frame of the source, and the period of the wave be \(t'\) in the rest frame, where \(dt' = 1/v'\).
In the time between emission of two wavefronts, $\Delta t$, the source moves a distance $l = v \Delta t$, and $d = v \Delta t \cos \theta$.

In the observer's frame, the time elapsed between emission of the wave fronts is $\Delta t' = \Delta t - \Delta t' = \Delta t' (1 - \beta \cos \theta)$.

The difference in arrival times of the wavefronts is $\Delta t A$, which is less than $\Delta t$ because the second wavefront has a shorter distance to travel:

$$\Delta t A = \Delta t - \frac{d}{c} = \Delta t - \frac{v \Delta t \cos \theta}{c} = \frac{\Delta t'}{(1 - \beta \cos \theta)}$$

The observed frequency is:

$$v = \frac{1}{\Delta t A} = \sqrt{\frac{1}{\Delta t'} (1 - \beta \cos \theta)}$$

Doppler effect has two terms that depend on $v/c$:

1. First-order, or longitudinal Doppler shift:
   $$\frac{1}{1 - \beta \cos \theta}$$ (red or blue depending on $\theta$)

2. Second-order, or transverse Doppler shift:
   $$\sqrt{1 - \beta^2}$$ (is always a redshift)

If $v < c$ then 1) $v/v' \approx (1 + \beta \cos \theta)$

2) can be neglected $\approx 1 - \frac{1}{2} \beta^2$

The Doppler formula is a mixed transformation since it has primed frequency but unprimed angle on the right side. A pure transformation can be written:

$$v' = v \sqrt{1 - \beta^2 (1 - \beta \cos \theta)}$$ or $$v = v' \frac{1}{\sqrt{1 - \beta^2 (1 - \beta \cos \theta)}}$$
The space-time interval between two events, defined as

\[ ds^2 = -(c \, dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = -(c \, dt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \]

is a Lorentz invariant (independent of reference frame).

If \( ds^2 < 0 \), the interval is *time-like* and the events can be causally related.

If \( ds^2 > 0 \), the interval is *space-like* and the events are not causally connected.

If \( ds^2 = 0 \), the interval is **null** or **light-like**.

The space-time interval is related to the *proper time*, \( d\tau \), which is the time elapsed between two events that occur at the same place (in the primed frame, as usual):

\[ c^2 d\tau^2 = (c \, dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = (c \, dt')^2 \]

Then \( d\tau = dt'/c \) and proper time is the minimum time interval between two events.

If the interval is space-like, i.e., there is no such frame, then proper time is undefined.

Proper time \( \tau \) and speed of light \( c \) are Lorentz invariants.

**Four-Vectors** - a four dimensional quantity that transforms according to the Lorentz transformations is called a four-vector. The scalar product of two four-vectors is a Lorentz invariant.

\[
\begin{pmatrix}
  ct' \\
  x' \\
  y'
\end{pmatrix}
= \begin{pmatrix}
  \gamma & -\beta & 0 & 0 \\
  -\beta & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  ct \\
  x \\
  y
\end{pmatrix}
\]

\[
\gamma^M = \sum_{i=0}^{3} \gamma_i \gamma^{i+1}
\]
In general, a four vector is of the form \((\gamma_0, \gamma_1, \gamma_2, \gamma_3)\). The scalar product is \(\mathbf{\cdot} \mathbf{\cdot} = -\gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2\)
or\[\mathbf{\cdot} \mathbf{\cdot} = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \gamma_{\mu\nu} x^\mu x^\nu = \gamma_{\mu\nu} x^\mu x^\nu\]
where \(\gamma_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\) is the Minkowski metric

Four-position is \(\mathbf{x} = (ct, x_1, x_2, x_3)\) \(x^\mu = \gamma_{\mu\nu} x^\nu\)

Four-velocity can be defined by differentiating 4-position
by proper time \(d\tau\) \((d\tau = dt/c)\)

\[U^\mu = \frac{dx^\mu}{d\tau} = (ct, x_1, x_2, x_3)\]

Note that \(U^\mu U_\mu = -c^2 \gamma_0^2 + \gamma_0^2 u^2 = -c^2 \gamma_0^2 \left(1 - u^2/c^2\right) = -c^2\)

Four-momentum multiply four-velocity by rest mass \(m_0\).

\[p^\mu = m_0 U^\mu\]

components \(p^0 = m_0 \gamma_0 c = E/c\)

\((1, 2, 3)\) \(\mathbf{p} = m_0 \gamma \mathbf{u}\) (relativistic 3-momentum)

\[p^\mu p_\mu = -\frac{E^2}{c^2} + p^2 = -m_0^2 c^2\]

\[E^2 = p^2 c^2 + m_0^2 c^4\]
Fun with 4-vectors

\(4-\text{vector} \quad \text{Symbol} \quad \text{Component} \quad \text{Component} \quad \text{Invariant} \)

\[
\begin{array}{cccc}
4-\text{position} & \chi^\mu & c \tau & - (c^2 \chi^2 + \chi^2 c^4 + \tau^2) = -c^2 \\
4-\text{velocity} & U^\mu & \gamma U^\mu c & -c^2 \\
4-\text{momentum} & p^\mu & m_0 \gamma U^\mu c & - m_0^2 c^2 \\
4-\text{current} & j^\mu & e \rho_0 U^\mu & - e^2 c^2 \\
4-\text{potential} & A^\mu & \Phi & \\
\end{array}
\]

\[\gamma = \sqrt{1 - \frac{u^2}{c^2}}\]

\[\rho^0 = m_0 U^0 = \frac{m_0 c}{\sqrt{1 - u^2/c^2}} = m_0 c + \frac{1}{2} m_0 u^2 + \ldots\]

\[\gamma c \frac{\rho^0}{2} = m_0 c^2 + \frac{1}{2} m_0 u^2 + \ldots = E \quad \text{total energy (relativistic)}\]

\[\rho_{1,2,3} = \gamma m_0 \frac{U}{u} \quad \text{relativistic momentum} \quad \vec{p} = \gamma m_0 \frac{U}{u}\]

\[\rho^\mu \rho_\mu = - \frac{E}{c^2} + \vec{p} \cdot \vec{p}\]

\[\text{In the frame in which } \vec{p} = 0 \quad \rho_\mu \rho^\mu = - m_0^2 c^2\]

\[\text{So } \rho_\mu \rho^\mu = \rho_\mu \rho^\mu = - m_0^2 c^2\]

\[E^2 = c^2 p^2 + m_0^2 c^4\]

\[\text{where } p = \gamma m_0 \frac{U}{u}\]

\[\text{Lorentz transformations } \rho_\mu = \Lambda^\nu_\mu \rho^\nu \quad \text{for energy/momentum}\]

\[\text{We may write } \rho' = \left(\frac{E'}{c'}, \vec{p}'\right) \quad \rho = \left(\frac{E}{c}, \vec{p}\right)\]

\[\Lambda = \begin{pmatrix} \gamma & -\beta \gamma & 0 \gamma & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\]

\[\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \quad \beta = \frac{u}{c}\]

\[\text{Then } E'/c' = \gamma E/c - \gamma \beta p_x\]

\[p'_x = \gamma \beta E/c + \gamma p_x\]

\[p'_y = p_y\]

\[p'_z = p_z\]

\[E' = \gamma (E - \gamma \beta p_x)\]

\[p'_x = \gamma (p_x - \frac{\gamma}{c} E)\]
A photon has zero rest mass, so its $P\cdot P^\mu = 0$.

Note that conservation of energy $E$ only holds in its relativistic form.

Also $\vec{F} = m\vec{a}$ does not hold in relativity. You have to define a "four-force" which is equal to $dP^\mu/d\tau$, where $d\tau$ is the proper time.

$$F^\mu = m_0 a^\mu = \frac{dP^\mu}{d\tau}$$

Four vectors can be used to derive interesting results such as the Doppler shift and the aberration of light formula.

Note that the 4-momentum for a photon can be written as

$$P = \left( \frac{E/c}{\bar{P}} \right) = \left( \frac{h\nu/c}{h\mu/(2\pi)} \right)$$

because $E = h\nu = \hbar\nu$

$$P = \frac{h\nu}{c} = \frac{h\nu}{2\pi}$$

So the quantity $K^\mu = (\nu/c)$ is a 4-vector.

Now take the scalar product of $U^\mu K^\mu$:

$$\gamma u \cdot \omega/c + \gamma u \cdot \bar{\omega}$$

$$\Rightarrow \gamma u (\omega - \bar{\omega} \cdot \bar{K}) = \gamma u' (\omega' - \bar{\omega}' \cdot \bar{K}')$$

In the rest frame of the observer $\bar{u}' = 0$, $\gamma u' = 1$

So $\gamma (\omega - \bar{\omega} \cdot \bar{K}) = \omega'$

but $c = \omega/k$ so $\gamma (\omega - \omega/k \bar{\omega} \cdot \bar{K}) = \omega'$

$$\gamma \omega \left( 1 - \frac{\bar{u} \cdot \bar{K}}{c} \right) = \omega'$$

$$\gamma \omega \left( 1 - \frac{\bar{\omega} \cdot \bar{K}}{c} \right) = \omega'$$

$$\gamma \sqrt{1 - \beta \omega} = \omega'$$

**Doppler shift**
Another method of deriving the doppler shift is to take the 0th component of the lorentz transformation of $k^\mu$

$$k^0 = \gamma k^0 - \gamma \beta k^1$$
$$k^0 = \frac{\omega}{c}$$
$$k^1 = (\frac{\omega}{c})\cos \theta$$

$$\frac{\omega'}{c} = \gamma \left[ \frac{\omega}{c} - \beta (\frac{\omega}{c}) \cos \theta \right]$$
$$\frac{\omega'}{c} = (\frac{\omega}{c}) \sin \theta$$

$$\omega' = \gamma \omega (1 - \beta \cos \theta)$$ which is the doppler shift

Since the inverse transformation is $\omega = \gamma \omega' (1 + \beta \cos \theta')$, we can multiply these together

$$\omega \omega' = \gamma^2 \omega \omega' (1 - \beta \cos \theta)(1 + \beta \cos \theta')$$

Solve for $\cos \theta$:

$$\frac{\gamma^2}{\beta} = 1 - \beta \cos \theta + \beta \cos \theta' - \beta^2 \cos \theta \cos \theta'$$

$$\frac{\gamma^2}{\beta} = 1 + \beta \cos \theta' - \beta (1 + \beta \cos \theta') \cos \theta$$

$$\cos \theta = \frac{1 - \beta \cos \theta'}{1 - \beta (1 + \beta \cos \theta') \cos \theta'} = -\frac{\beta (\beta + \cos \theta)}{\beta (1 + \beta \cos \theta') - \beta (1 + \beta \cos \theta') \cos \theta'}$$

$$\cos \theta = \frac{\beta + \cos \theta'}{1 + \beta \cos \theta'}$$

Same as before

Another 4-vector is $J^\mu = (c\rho, \frac{\vec{\mu}}{c})$

$\vec{J} = \rho \vec{\mu}$ where $\rho = \frac{d\rho}{dv}$ (charge density)

In the rest frame of the charge, which we choose to be the primed frame, we can defined a proper charge density (note that $\alpha$ is an invariant)

$$\rho_0 = \frac{d\rho}{dv}, \text{ (proper charge density)}$$
But the \( dV = dx' dy' dz' \) is not equal to \( dV' = dx'dy'dz' \) because the length in the \( x \)-direction is Lorentz contracted.

\[
dx = \frac{dx'}{\gamma} \quad \text{so} \quad dV' = \gamma dx dy dz = \gamma dV
\]

So \( \rho_0 = \frac{\gamma \rho_0}{\sqrt{1-u^2/c^2}} \frac{d\varphi}{dV} \) is the proper charge density, which is the charge density measured in \( S' \) (i.e. \( \rho' = \rho_0 \)).

(Another way of writing this is: \( \gamma \rho_0 = \rho' \)).

So \( \rho = \frac{\gamma \rho_0}{\sqrt{1-u^2/c^2}} \) and \( \dot{\mathbf{J}} = \frac{\rho_0 \mathbf{u}}{\sqrt{1-u^2/c^2}} \) are the components of a 4-vector which is just like \( \mathbf{U}^\mu \),

\[
(\text{so } \rho = \rho_0 \mathbf{U}^0, \quad \dot{\mathbf{J}} = \rho_0 \mathbf{U}^\mu)
\]

[Remember that we always denote a 4-vector with a superscript or subscript, e.g. \( \mathbf{J}^\mu \), while we denote its spatial components \( (1,2,3) \) with an arrow, e.g. \( \mathbf{J} \).]

Now we can show that the equations of electrodynamics are “covariant,” that is, they are laws which are correct in any frame of reference. For example:

**The Continuity Equation**

\[
\nabla \cdot \dot{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0
\]

Since \( \nabla \cdot \dot{\mathbf{J}} = \sum_{\mu=1}^{3} \frac{\partial J^\mu}{\partial x^\mu} \) and \( \frac{\partial \rho}{\partial t} = \frac{1}{c} \frac{\partial (\rho_0 \mathbf{u}^0)}{\partial t} = \frac{\partial \rho_0}{\partial t} \mathbf{u}^0 \)

We can write the tensor equation

\[
\frac{\partial J^\mu}{\partial x^\mu} = 0
\]

Or abbreviating \( \dot{\mathbf{A}} \equiv \frac{\partial \mathbf{A}}{\partial t} \)
the “4-gradient” \( \frac{\partial}{\partial x^\mu} \)

Since \( x^\mu \) is a contravariant vector, \( \frac{\partial}{\partial x^\mu} \) is a covariant operator

\[
\frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)
\]

Note \( \frac{\partial}{\partial x^\mu} = \eta^\nu_\mu \frac{\partial}{\partial x^\nu} \)