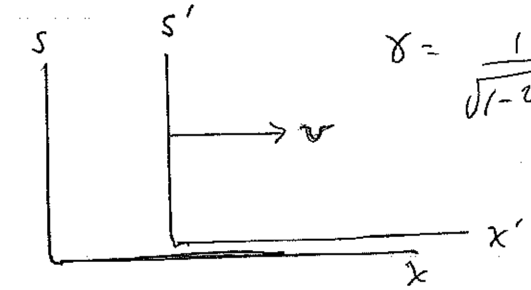
Special Relativity

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Lorentz Transformations



$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^2)$$

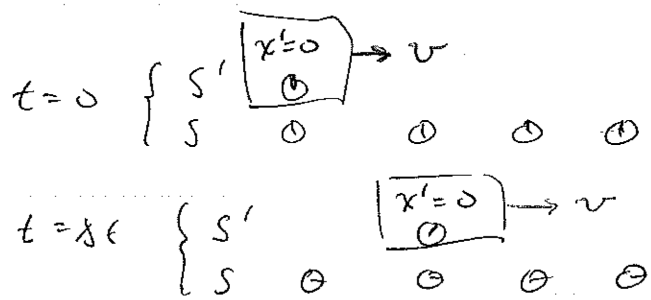
Inverse Transformations

(exchange prime and unprimed, change sign of v)

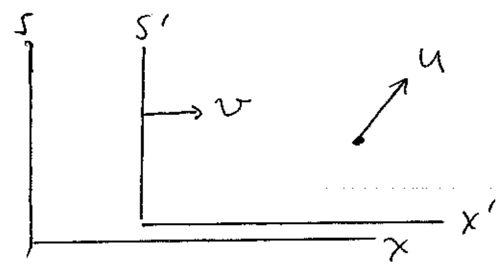
$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + vx'/c^2)$$

Time Dilation

$$\Delta t = \gamma \Delta t'$$



Transformation of Velocities



$$dx' = \gamma(dx - v\,dt)$$
$$dy' = dy$$
$$dz' = dz$$
$$dt' = \gamma(dt - v\,dx/c^2)$$

$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v\,dx}{c^2}} = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v\,dx}{c^2}\right)} = \frac{u_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}$$

$$u_z' = \frac{u_z}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}$$

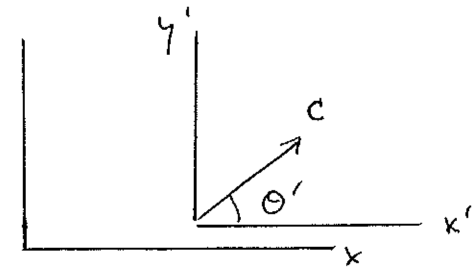
Inverse Transformation of Velocities (exchange prime and unprimed, change sign of v)

$$u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}}$$

$$u_y = \frac{u_y'}{\gamma\left(1 + \frac{v u_x'}{c^2}\right)} \qquad u_z = \frac{u_z'}{\gamma\left(1 + \frac{v u_x'}{c^2}\right)}$$

Application of Transformation of Velocities to Aberration of Light

A ray of light has



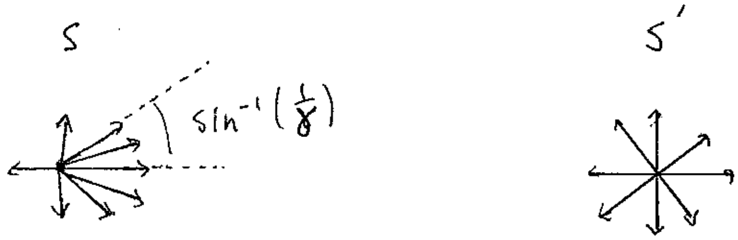
$$u_x' = c\cos\theta' \quad , \quad u_y' = c\sin\theta'$$

$$u_x = c\cos\theta \quad , \quad u_y = c\sin\theta$$

$$c\cos\theta = \frac{c\cos\theta' + v}{1 + \frac{v}{c}\cos\theta'} \qquad c\sin\theta = \frac{c\sin\theta'}{\gamma\left(1 + \frac{v}{c}\cos\theta'\right)}$$

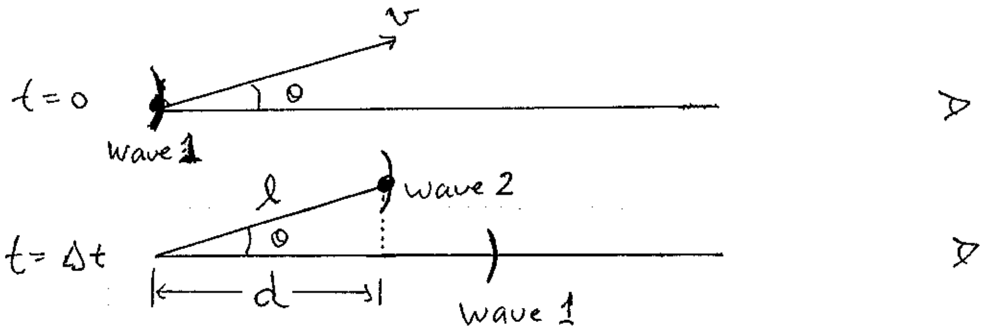
$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')}$$

This shows that $\theta < \theta'$. In the case that $\theta' = \pi/2$, it is useful to note that $\sin\theta = 1/\gamma$. In the limit $\gamma \gg 1$, $\theta \simeq 1/\gamma$. If the source is an isotropic emitter, half the radiation is beamed into an angle $< 1/\gamma$



Doppler effect - involves the arrival rate of wavefronts emitted by a moving source

Let the frequency of radiation be ν' in the rest frame of the source, and the period of the wave $\Delta t'$ in the rest frame, where $\Delta t' = 1/\nu'$



In the time between emission of two wavefronts, Δt, the source moves a distance $l = v\Delta t$, and $d = v\Delta t \cos\theta$

In the observer's frame, the time elapsed between <u>emission</u> of the wave fronts is $\Delta t = \gamma\Delta t' = \gamma/\nu'$.

The difference in arrival times of the wave fronts is Δt_A, which is less than Δt because the second wave front has a shorter distance to travel

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t - \frac{v\Delta t\cos\theta}{c} = \gamma\Delta t'(1-\beta\cos\theta)$$

The observed frequency $\boxed{\nu = \frac{1}{\Delta t_A} = \nu'\left[\gamma(1-\beta\cos\theta)\right]^{-1}}$

Doppler effect has two terms that depend on v/c

1) First-order, or <u>longitudinal</u> Doppler shift

$$\frac{1}{1-\beta\cos\theta} \qquad \text{(red or blue depending on } \theta\text{)}$$

2) Second-order, or <u>transverse</u> Doppler shift

$$\sqrt{1-\beta^2} \qquad \text{(is always a redshift)}$$

If $v \ll c$ then 1) $\nu/\nu' \approx (1+\beta\cos\theta)$

2) can be neglected $\approx 1 - \frac{1}{2}\beta^2$

The Doppler formula is a <u>mixed transformation</u> since it has primed frequency but unprimed angle on the right side. A pure transformation can be written:

$$\nu' = \nu\gamma(1-\beta\cos\theta) \qquad \text{or} \qquad \nu = \nu'\gamma(1+\beta\cos\theta')$$

The space-time interval between two events, defined as

(t_2, x_2, y_2, z_2)

$$ds^2 = -(c\,dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$
$$= -(c\,dt')^2 + (dx')^2 + (dy')^2 + (dz')^2$$

(t_1, x_1, y_1, z_1)

is a <u>Lorentz invariant</u> (independent of reference frame)

if $ds^2 < 0$, the interval is <u>time-like</u> and the events can be causally related.

if $ds^2 > 0$, the interval is <u>space-like</u> and the events are not causally connected

if $ds^2 = 0$, the interval is <u>null</u> or <u>light-like</u>

The space-time interval is related to the <u>proper time</u>, $d\tau$, which is the time elapsed between two events that occur at the same place (in the primed frame, as usual)

$$c^2 d\tau^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = (c\,dt')^2$$

Then $d\tau = dt' = dt/\gamma$ and proper time is the <u>minimum</u> time interval between two events.

If the interval is space-like, i.e., there is no such frame, then proper time is undefined.

Proper time τ and speed of light c are <u>Lorentz invariants</u>.

<u>Four-Vectors</u> – a four dimensional quantity that transforms according to the Lorentz transformations is called a four-vector. The scalar product of two four vectors is a Lorentz invariant

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

row #

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

column #

In general, a four vector is of the form (x_0, x_1, x_2, x_3).

The scalar product is $x \cdot x = -x_0^2 + x_1^2 + x_2^2 + x_3^2$

or $\quad x \cdot x = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu \equiv \eta_{\mu\nu} x^\mu x^\nu$

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the Minkowski metric

$x_\mu = \eta_{\mu\nu} x^\nu$

Four-position is $x^\mu \equiv (ct, x, y, z) \qquad x_\mu \equiv (-ct, x, y, z)$

Four-velocity can be defined by differentiating 4-position by proper time $d\tau$ $\quad (d\tau = dt/\gamma)$

$$u^\mu = \frac{dx^\mu}{d\tau} = (c\gamma_u, \gamma_u u_x, \gamma_u u_y, \gamma_u u_z)$$

Note that $u^\mu \cdot u_\mu = -\gamma_u^2 c^2 + \gamma_u^2 u^2 = -\gamma_u^2 c^2 \left(1 - \frac{u^2}{c^2}\right) = -c^2$

Four-momentum multiply four-velocity by rest mass m_0.

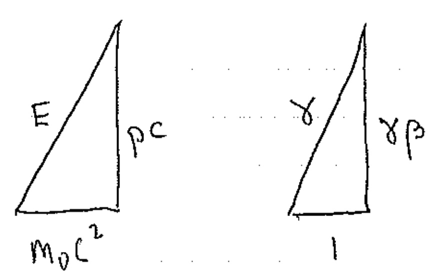
$$p^\mu = m_0 u^\mu$$

components $\quad p^0 = m_0 \gamma c = E/c$

$(1,2,3) \quad \vec{p} = m_0 \gamma \vec{u}$ (relativistic 3-momentum)

$$p^\mu p_\mu = -\frac{E^2}{c^2} + p^2 = -m_0^2 c^2$$

$$\Rightarrow \boxed{E^2 = p^2c^2 + m_0^2 c^4}$$



Fun with 4-vectors

4-vector	Symbol	Component 0	Component 1,2,3	Invariant
4-position	x^μ	ct	$\vec{x}$	$-(ct)^2 + x^2 + y^2 + z^2 = -(c\tau)^2$
4-velocity	U^μ	$\gamma_u c$	$\gamma_u \vec{u}$	$-c^2$
4-momentum	p^μ	$m_0 \gamma_u c$	$m_0 \gamma_u \vec{u} = m_0 U^{1,2,3}$	$-m_0^2 c^2$
4-current	J^μ	$c\rho = \rho_0 U^0$	$\vec{J} = \rho_0 U^{1,2,3}$	$-\rho_0^2 c^2$
4-vector potential	A^μ	Φ	$\vec{A}$	

$$\gamma_u = \sqrt{1 - u^2/c^2}$$

rest mass (also a Lorentz invariant)

$$p^0 = m_0 U^0 = \frac{m_0 c}{\sqrt{1-u^2/c^2}} = m_0 c + \frac{1}{2} m_0 \frac{u^2}{c} + \dots$$

$$\therefore c\, p^0 = \underset{\uparrow \text{ rest Energy}}{m_0 c^2} + \underbrace{\frac{1}{2} m_0 u^2 + \dots}_{\text{kinetic Energy}} = E \quad \text{total energy (relativistic)}$$

$$p^{1,2,3} = \gamma_u m_0 \vec{u} \qquad \text{relativistic momentum } \vec{p} = \gamma_u m_0 \vec{u}$$

$$p_\mu p^\mu = -(E/c)^2 + \vec{p}\cdot\vec{p}$$

In the frame in which $\vec{p} = 0$, $\quad p'_\mu p'^\mu = -m_0^2 c^2$

So
$$p'_\mu p'^\mu = p_\mu p^\mu$$
$$-m_0^2 c^2 = -(E/c)^2 + p^2$$
$$E^2 = c^2 p^2 + m_0^2 c^4 \qquad \text{where } p = \gamma_u m_0 u$$

Lorentz transformations $\quad p'^\mu = \Lambda^\mu{}_\nu p^\nu \quad$ for energy/momentum

We may write $\quad p' = \begin{pmatrix} E'/c \\ \vec{p}' \end{pmatrix} \qquad p = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \beta = \frac{v}{c}$$

Then
$$E'/c = \gamma E/c - \gamma\beta p_x$$
$$p'_x = -\gamma\beta E/c + \gamma p_x$$
$$p'_y = p_y$$
$$p'_z = p_z$$

$$E' = \gamma(E - v p_x)$$
$$p'_x = \gamma\left(p_x - \frac{v}{c^2} E\right)$$

A photon has zero rest mass, so its $p_\mu p^\mu = 0$

Note that conservation of energy E only holds in its relativistic form.

Also $\vec{F} = m\vec{a}$ does not hold in relativity. You have to define a "four-force" which is equal to $dp^\mu/d\tau$, where $d\tau$ is the proper time

$$F^\mu = m_0 a^\mu = \frac{dp^\mu}{d\tau}$$

Four vectors can be used to derive interesting results such as the Doppler shift and the abberation of light formula. Note that the 4-momentum for a photon can be written as

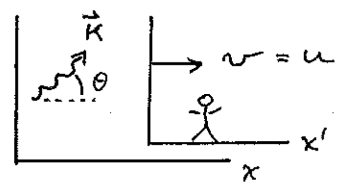
$$P = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \begin{pmatrix} h\nu/c \\ h\vec{k}/(2\pi) \end{pmatrix}$$

because $E = h\nu = \hbar\omega$

$p = h/\lambda = \frac{h k}{2\pi} = \hbar k$

So the quantity $k^\mu = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}$ is a 4-vector

Now take the scalar product of $u_\mu k^\mu$



$$-\gamma_u c\,\omega/c + \gamma_u \vec{u}\cdot\vec{k}$$

$$\Rightarrow \gamma_u(\omega - \vec{u}\cdot\vec{k}) = \gamma_{u'}(\omega' - \vec{u}'\cdot\vec{k}')$$

In the rest frame of the observer $\vec{u}' = 0$, $\gamma_{u'} = 1$

So $\gamma(\omega - \vec{u}\cdot\vec{k}) = \omega'$

But $c = \omega/k$ so $\gamma\left(\omega - \frac{\omega/k}{c}\,\vec{u}\cdot\vec{k}\right) = \omega'$

$$\gamma\omega\left(1 - \frac{\vec{u}}{c}\cdot\frac{\vec{k}}{k}\right) = \omega'$$

$$\gamma\omega(1 - \vec{\beta}\cdot\hat{k}) = \omega'$$

$$\boxed{\gamma\nu(1 - \beta\cos\theta) = \nu'}$$ Doppler shift

Another method of Deriving the Doppler shift is to take the 0^{th} component of the Lorentz transformation of k^μ

$$k'^0 = \gamma k^0 - \gamma\beta k^1 \qquad k^0 = \omega/c$$

$$\omega'/c = \gamma[\omega/c - \beta(\omega/c)\cos\theta] \qquad k^1 = (\omega/c)\cos\theta, \quad k^2 = (\omega/c)\sin\theta$$

$$\omega' = \gamma\omega(1-\beta\cos\theta)$$ which is the Doppler shift

Since the inverse transformation is $\omega = \gamma\omega'(1+\beta\cos\theta')$, we can multiply these together

$$\omega'\omega = \gamma^2\omega\omega'(1-\beta\cos\theta)(1+\beta\cos\theta')$$

Solve for $\cos\theta$:

$$\frac{1}{\gamma^2} = 1 - \beta\cos\theta + \beta\cos\theta' - \beta^2\cos\theta\cos\theta'$$

$$\frac{1}{\gamma^2} = 1 + \beta\cos\theta' - \beta(1+\beta\cos\theta')\cos\theta$$

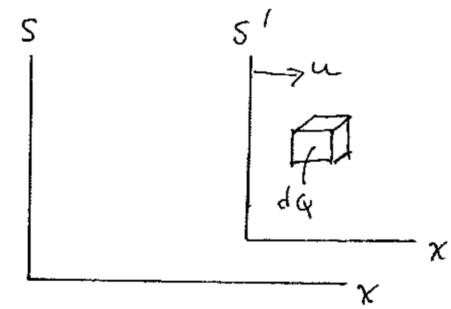
$$\cos\theta = \frac{\frac{1}{\gamma^2} - 1 - \beta\cos\theta'}{-\beta(1+\beta\cos\theta')} = \frac{-\beta(\beta+\cos\theta')}{-\beta(1+\beta\cos\theta')} = \frac{\beta+\cos\theta'}{1+\beta\cos\theta'}$$

Same as before

<u>Another 4-vector is $J^\mu = (c\rho, \vec{J})$</u>

$\vec{J} = \rho\vec{u}$ where $\rho = \frac{dQ}{dV}$ (charge density)

In the rest frame of the charge, which we choose to be the primed frame, we can defined a <u>proper charge density</u> (note that Q is an invariant)



$$\rho_0 = \frac{dQ}{dV'} \quad \text{(proper charge density)}$$

But the $dV = dx\,dy\,dz$ is not equal to $dV' = dx'dy'dz'$ because the length in the x-direction is Lorentz contracted.

$$dx = dx'/\gamma \quad \text{so} \quad dV' = \gamma\, dx\, dy\, dz = \gamma\, dV$$

So $\rho_0 = \sqrt{1-u^2/c^2}\,\frac{dQ}{dV}$ is the proper charge density, which is the charge density measured in S' (i.e. $\rho' = \rho_0$)

(Another way of writing this is: $\rho' = \sqrt{1-u^2/c^2}\,\rho$)

so $\rho = \frac{\rho_0}{\sqrt{1-u^2/c^2}}$ $\quad \vec{J} = \frac{\rho_0 \vec{u}}{\sqrt{1-u^2/c^2}}$ are the components of a 4-vector which is just like U^μ

$$\left(c\rho = \rho_0 U^0 \quad , \quad \vec{J} = \rho_0 U^{1,2,3} \right)$$

[Remember that we always denote a 4-vector with a superscript or subscript, e.g. J^μ, while we denote its spatial components (1,2,3) with an arrow, e.g. $\vec{J}$.]

Now we can show that the Equations of electrodynamics are "Covariant", that is, they are laws which are correct in any frame of reference. For example:

<u>The Continuity Equation</u>

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Since $\nabla \cdot \vec{J} = \sum_{\mu=1}^{3} \frac{\partial J^\mu}{\partial x^\mu}$ and $\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}(c\rho) = \frac{\partial J^0}{\partial x^0}$

We can write the tensor equation $\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$

Or abbreviating the "4-gradient" $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ $\quad \boxed{\partial_\mu J^\mu = 0}$

Since x^μ is a contravariant vector, $\frac{\partial}{\partial x^\mu}$ is a <u>covariant operator</u>

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Note $\frac{\partial}{\partial x_\mu} = \eta^{\mu\nu}\frac{\partial}{\partial x^\nu}$