Line Broadening and Curve of Growth

Doppler Broadening due to Maxwell-Boltzmann distribution in one dimension (v_z)

$$f(v_z)\,dv_z = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_z^2}{2kT}}\,dv_z \qquad m = \text{atom mass.}$$

Frequency shift $\frac{\nu-\nu_0}{\nu_0} = \frac{v_z}{c}$ so $v_z = \frac{c(\nu-\nu_0)}{\nu_0}$

Doppler width $\Delta\nu_D = \sqrt{\frac{2kT}{m}}\,\frac{\nu_0}{c}$ where $\sqrt{\frac{2kT}{m}} = 12.9\sqrt{\frac{T}{10^4}\frac{1}{A}}\ \frac{km}{s}$

Doppler line profile $\phi(\nu)\,d\nu = \frac{1}{\Delta\nu_D\sqrt{\pi}}\,e^{-\frac{(\nu-\nu_0)^2}{(\Delta\nu_D)^2}}\,d\nu$

Integrated Cross section $\int\sigma(\nu)\,d\nu = \int\phi(\nu)\,d\nu\ B_{12}\frac{h\nu_0}{4\pi} = \frac{\pi e^2}{m_e c} f_{12} = \sigma \quad \left[\frac{cm^2}{s}\right]$

Cross section at line center $\sigma(\nu_0) = \frac{B_{12}}{\Delta\nu_D\sqrt{\pi}}\,\frac{h\nu_0}{4\pi} = \frac{1}{\Delta\nu_D\sqrt{\pi}}\,\frac{\pi e^2}{m_e c} f_{12}$

$$\sigma(\nu_0) = 1.16\times10^{-14}\ \lambda_0\sqrt{\frac{A}{T}}\ f_{12} \quad [cm^2]$$

where λ_0 is in Angstroms and A is the atomic mass #

Lorentzian (natural) line profile combined with Doppler broadening

$$\sigma(\nu) = \frac{\pi e^2}{m_e c}\,\frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2}\,f_{12} = \sigma\,\phi(\nu)$$

To combine Doppler with natural line broadening, replace ν_0 with $\nu_0 + \nu_0 v_z/c$ and integrate over the M-B distribution

$$\phi(\nu) = \frac{\Gamma}{4\pi^2}\left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{\infty}\frac{e^{-mv_z^2/2kT}\,dv_z}{\left(\nu-\nu_0-\nu_0\frac{v_z}{c}\right)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

Change variables to $y^2 = \frac{m v_z^2}{2kT}$ $\quad dv_z = \sqrt{\frac{2kT}{m}}\, dy$

$$\phi(\nu) = \frac{\Gamma}{4\pi^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}\, dy}{\left(\frac{\nu-\nu_0}{\Delta\nu_D} - y\right)^2 + \left(\frac{\Gamma}{4\pi\Delta\nu_D}\right)^2} \left(\frac{1}{\Delta\nu_D}\right)^2$$

Define $u \equiv \frac{\nu - \nu_0}{\Delta\nu_D}$

$$a \equiv \frac{\Gamma}{4\pi\Delta\nu_D}$$ ($a \ll 1$ in astronomical situations)

$$\phi(\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D} \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}\, dy}{(u-y)^2 + a^2}$$

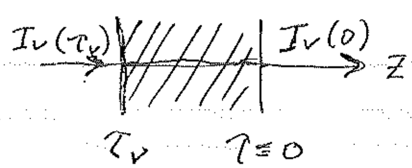
Voigt function $H(a,u) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}\, dy}{(u-y)^2 + a^2}$

Limiting expressions $H(a,u) \simeq e^{-u^2}$ for $u \ll 1$

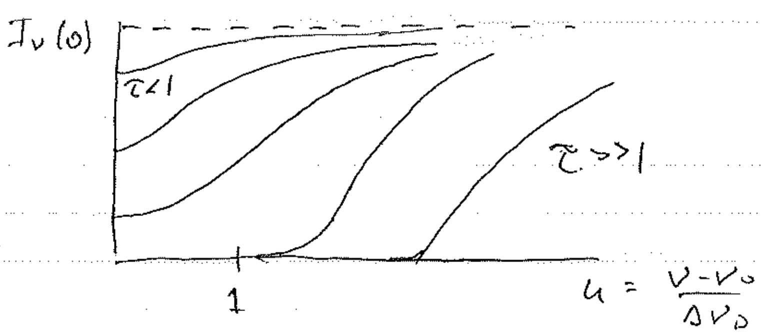
$$H(a,u) \simeq \frac{a}{\sqrt{\pi}\, u^2} \quad \text{for } u \gg 1$$

<u>Radiative transfer in line</u>

$$I_\nu(0) = I_\nu(\tau_\nu)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu\, e^{-\tau_\nu'}\, d\tau_\nu'$$

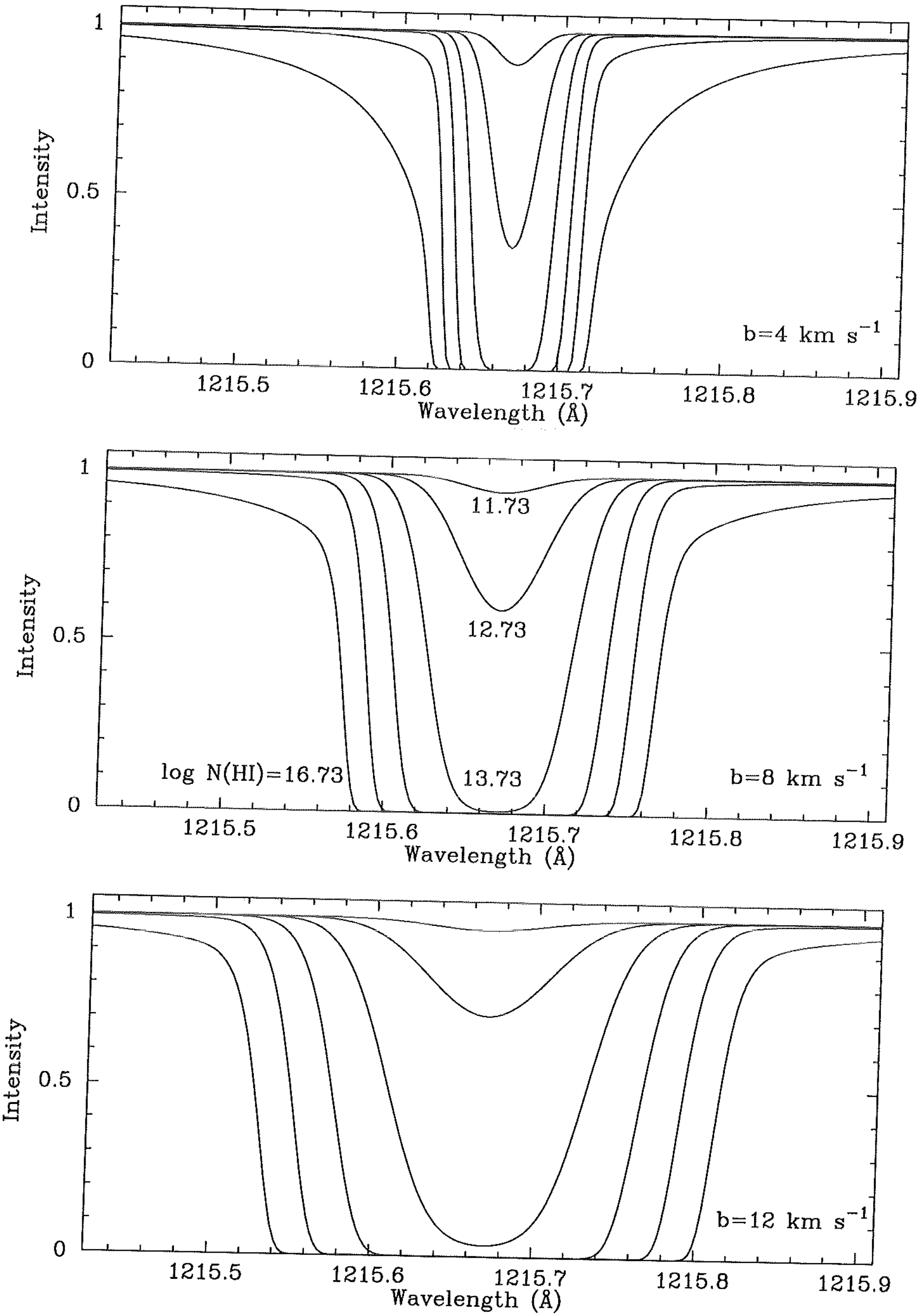


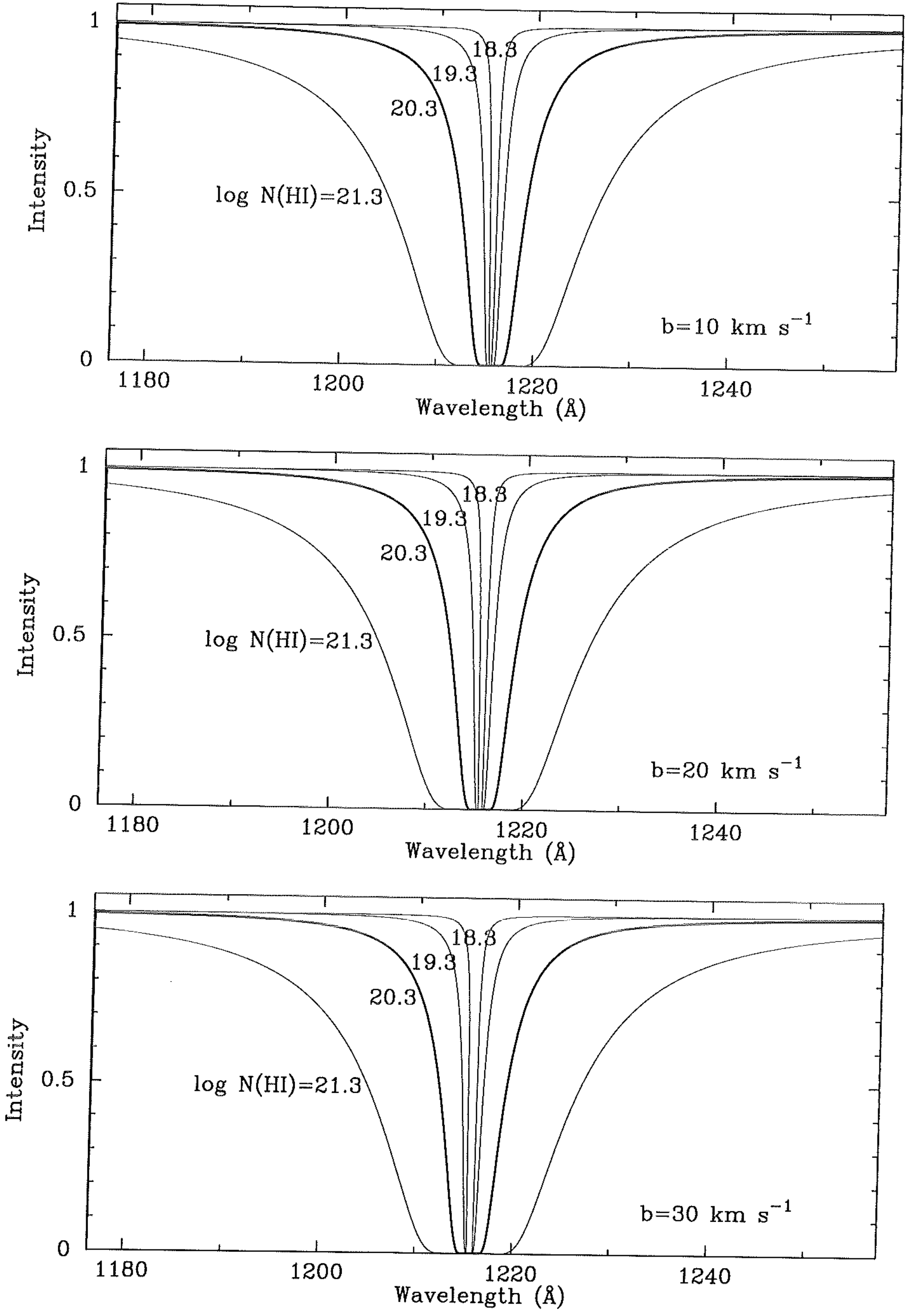
<u>Line Profiles</u> (in the case of pure absorption $S_\nu = 0$)



Line profiles as a function of increasing τ_ν

The curve of growth refers to the <u>equivalent width</u> of the line,


1
0.5
0
Intensity
b=4 km s^{-1}
1215.5
1215.6
1215.7
1215.8
1215.9
Wavelength (Å)
1
0.5
0
Intensity
11.73
12.73
13.73
log N(HI)=16.73
b=8 km s^{-1}
1215.5
1215.6
1215.7
1215.8
1215.9
Wavelength (Å)
1
0.5
0
Intensity
b=12 km s^{-1}
1215.5
1215.6
1215.7
1215.8
1215.9
Wavelength (Å)



18.3
19.3
20.3
log N(HI)=21.3
b=10 km s^{-1}
Intensity
Wavelength (Å)
18.3
19.3
20.3
log N(HI)=21.3
b=20 km s^{-1}
Intensity
Wavelength (Å)
18.3
19.3
20.3
log N(HI)=21.3
b=30 km s^{-1}
Intensity
Wavelength (Å)

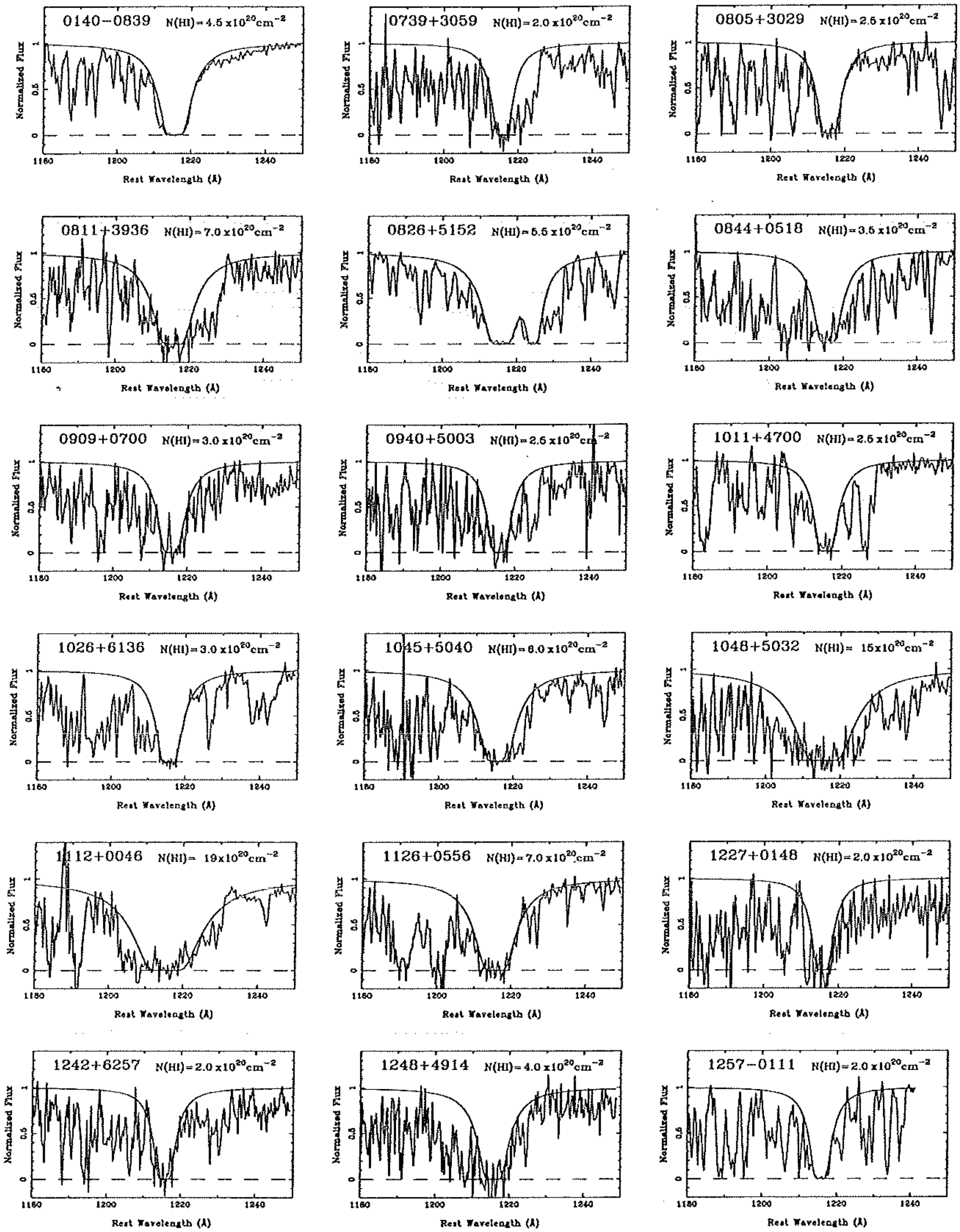
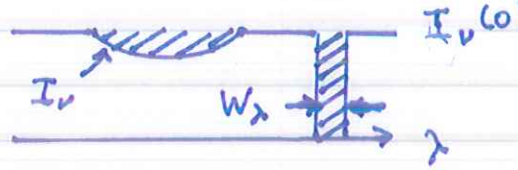


Figure 1. Theoretical damped Lyα profile fits to all 33 PDLAs. The fitted N(H I) is indicated at the top right of each plot. Table 2 gives the derived $z_{\rm abs}$ for each absorption system. For QSO 0826+5152, the fit is for the two absorbers, and the wavelength scale is in the rest frame of the blueward feature, which is the PDLA in this system. The HI column density of the redward absorber is 1.3×10^{20} cm^{-2}.

If the line profile can't be resolved, its easier to measure the Equivalent Width, W_λ.



$$W_\lambda = \int_0^\infty \frac{I_\nu(0) - I_\nu}{I_\nu(0)} d\lambda$$

$$= \int_0^\infty (1 - e^{-\tau_\nu}) d\lambda \quad \text{where } \tau_\nu = \int n_1 \sigma_\nu dz$$

$$= \frac{\lambda^2}{c} \int_0^\infty (1 - e^{-\tau_\nu}) d\nu \quad \text{where } d\nu = -\frac{c}{\lambda^2} d\lambda$$

Curve of Growth is the dependence of $\frac{W_\lambda}{\lambda}$ on the parameters $N_1 \lambda f_{12}$.
It has three parts, depending on the parameters

1. Linear part when $\tau_\nu \ll 1$ all across the line

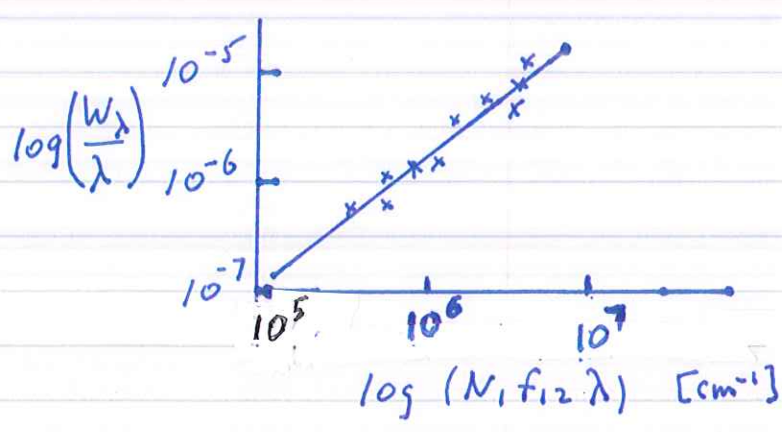
$$W_\lambda \approx \frac{\lambda^2}{c} \int_0^\infty \tau_\nu d\nu = \frac{\lambda^2}{c} \int n_1 dz \int_0^\infty \sigma_\nu d\nu$$

$$= \frac{\lambda^2}{c} N_1 \frac{\pi e^2}{m_e c} f_{12}$$

where N_1 is the column density [cm^{-2}] of a species in the lower level

$$\boxed{\frac{W_\lambda}{\lambda} = \frac{\pi e^2}{m_e c^2} (N_1 \lambda f_{12}) = 8.85 \times 10^{-13} N_1 \lambda f_{12}}$$

The linear part of the curve of growth is a universal function. All lines from any species will fall on the same curve.



The general curve of growth depends on the function $\phi(\nu)$ and therefore on the Doppler width $\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$

In the case of a Doppler profile:

$$\phi(\nu) = \frac{c}{\nu_0}\sqrt{\frac{m}{2kT}}\frac{1}{\sqrt{\pi}}\, e^{-(\nu-\nu_0)^2/\left(\frac{2kT}{m}\frac{\nu_0^2}{c^2}\right)}$$

Let $b \equiv \sqrt{\frac{2kT}{m}}$ "velocity width"

Then $\phi(\nu) = \frac{\lambda}{\sqrt{\pi}\, b}\, e^{-\left[\frac{c(\nu-\nu_0)}{\nu_0}\right]^2/b^2}$

$$\frac{W_\lambda}{\lambda} = \frac{\lambda}{c}\int_0^\infty (1-e^{-\tau_\nu})\,d\nu$$

Let $\tau_\nu = \tau_0 e^{-x^2}$ where $x = \frac{c}{\nu_0}\frac{(\nu-\nu_0)}{b}$

and $\tau_0 = \frac{\lambda}{\sqrt{\pi}\, b} N_1 \sigma = \frac{\lambda}{\sqrt{\pi}\, b} N_1 \frac{\pi e^2 f_{12}}{m_e c}$

Note: $\tau_0 = \frac{1.50\times10^{-2}}{b} N_1 f_{12} \lambda$

$$\frac{W_\lambda}{\lambda} = \frac{\lambda}{c}\int_{-\infty}^{\infty}\left[1-e^{-(\tau_0 e^{-x^2})}\right]dx\,\frac{\nu_0 b}{c}$$

$$= \frac{2b}{c}\int_0^\infty\left[1-e^{-(\tau_0 e^{-x^2})}\right]dx \quad \text{(changed limits of integral)}$$

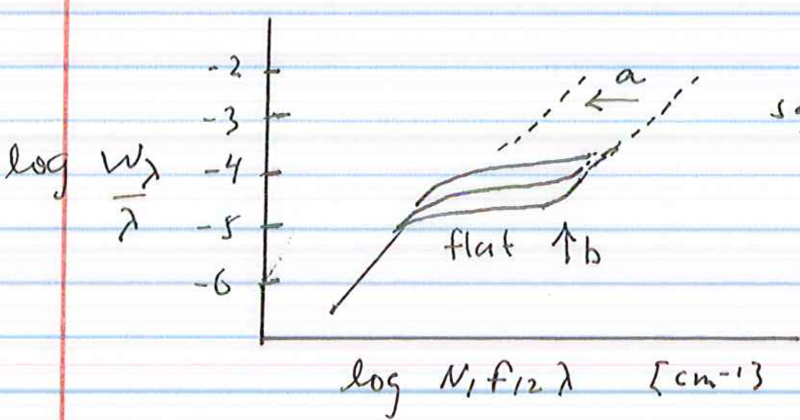
(Note: $-\infty < x < \infty$ while $0 < \nu < \infty$)

In the limit of large τ_0, the integrand is $\approx 1/2$ for $\tau_0 e^{-x^2} \approx \ln 2$, or $x = \sqrt{\ln(\tau_0/\ln 2)}$

2. $$\boxed{\frac{W_\lambda}{\lambda} \simeq \frac{2b}{c}\sqrt{\ln(\tau_0/\ln 2)}}$$

Flat, or Saturated

This is called the flat part of the curve of growth, where lines are saturated.



3. Square-root part of the curve of growth

For very strong lines the damping wings are important and the Voigt profile must be used

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}} H(a,u) = \frac{\lambda}{\sqrt{\pi}\, b} H(a,u)$$

$$\simeq \frac{\lambda}{\sqrt{\pi}\, b} \frac{a}{\sqrt{\pi}\, u^2} \quad \text{in the limit } u \gg 1$$

$$\tau_\nu = \frac{\lambda}{\sqrt{\pi}\, b} \frac{a}{\sqrt{\pi}\, u^2} N_1 \sigma_{12} \qquad \sigma_{12} = \frac{\pi e^2}{m_e c} f_{12}$$

In this limit, the integral $\frac{W_\lambda}{\lambda} = \frac{\lambda}{c}\int_0^\infty (1-e^{-\tau_\nu})\,d\nu$

is $$\boxed{\frac{W_\lambda}{\lambda} = \frac{2}{c}\left(\lambda^2 N_1 \sigma_{12} \frac{\Gamma}{4\pi}\right)^{1/2}} \qquad \Gamma = A_{21}$$