# Classical Oscillator Model of Line Emission

Harmonic oscillator: $F = -kx$ $(m\ddot{x} = -kx)$ has solution $x = Ae^{i\omega_0 t}$, $\omega_0 = \sqrt{k/m}$ natural frequency

Damped harmonic oscillator: Assume a radiation reaction force proportional to velocity $\dot{x}$ with proportionality constant $-m\Gamma$

$$m\ddot{x} = -kx - m\Gamma\dot{x} \qquad (1)$$

Assume solution $x = Ae^{\alpha t}$ and substitute into (1)

Then $m\alpha^2 x + kx + m\Gamma\alpha x = 0$

$$\alpha^2 + \omega_0^2 + \Gamma\alpha = 0$$

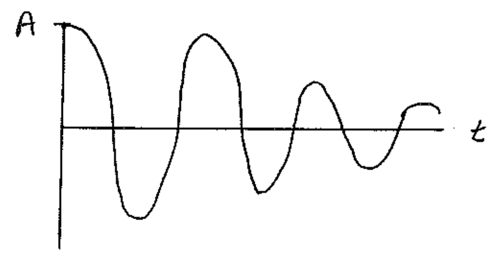
$$\alpha = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\omega_0^2}}{2}$$

$$\alpha = -\frac{\Gamma}{2} \pm \omega_0\sqrt{\frac{\Gamma^2}{(2\omega_0)^2} - 1}$$

On the next page we will show that $\Gamma \ll \omega_0$

$$\alpha \approx -\frac{\Gamma}{2} \pm i\omega_0$$

Solution $x(t) = Ae^{-\Gamma t/2}e^{\pm i\omega_0 t}$




For definiteness specify initial conditions $x = A$, $\dot{x} = 0$, at $t = 0$

$$x(t) = Ae^{-\Gamma t/2}\cos(\omega_0 t) = \frac{1}{2}A\left[e^{-\Gamma t/2 + i\omega_0 t} + e^{-\Gamma t/2 - i\omega_0 t}\right]$$

Fourier transform of $x$: $\hat{x}(\omega) = \frac{1}{2\pi}\int_0^\infty x(t)e^{i\omega t}\,dt$

$$\hat{x}(\omega) = \frac{A}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega + \omega_0)} + \frac{1}{\Gamma/2 - i(\omega - \omega_0)}\right]$$

Only positive frequency is relevant; $\hat{x}(\omega)$ becomes large only when $\omega \approx \omega_0$.

$$\hat{x}(\omega) \approx \frac{A}{4\pi} \frac{1}{\Gamma/2 - i(\omega-\omega_0)}$$

$$|\hat{x}(\omega)|^2 = \left(\frac{A}{4\pi}\right)^2 \frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

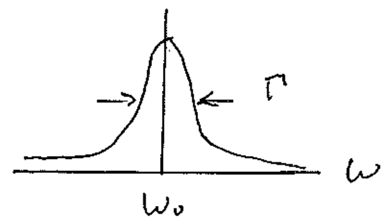
$d = ex$
dipole moment
(Fourier transform)
$\downarrow$

Energy per unit frequency is $\dfrac{dW}{d\omega} = \dfrac{8\pi}{3c^3}\,\omega^4\,|\hat{d}(\omega)|^2$

$\hat{d}(\omega) = e\,\hat{x}(\omega)$

$$\frac{dW}{d\omega} = \frac{8\pi}{3}\frac{\omega^4}{c^3} e^2 \left(\frac{A}{4\pi}\right)^2 \frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

$$\frac{dW}{d\omega} = \left(\frac{1}{2}kA^2\right)\underbrace{\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}}$$




Lorentz profile
(or damping profile)

$\Gamma$ = "classical line width"

Radiation reaction force $F_{rad}$ is the damping force

Power radiated is $P = \frac{2}{3}\frac{e^2}{c^3}(\ddot{x})^2$

$$P = -F_{rad}\cdot\dot{x} = \frac{2}{3}\frac{e^2}{c^3}\ddot{x}^2$$

If $x \propto \cos(\omega_0 t)$, $\quad F_{rad} = -\frac{2}{3}\frac{e^2}{c^3}\omega_0^2\dot{x}$

Since we define $\Gamma$ as $F_{rad} = -m\Gamma\dot{x}$, $\quad \boxed{\Gamma = \frac{2}{3}\frac{e^2}{c^3 m}\omega_0^2}$

For Lyman $\alpha$ radiation $\lambda_0 = 1215\,\text{Å}$, $\Gamma = 1.5\times10^9\ s^{-1}$

Since $\lambda = \frac{2\pi c}{\omega}$, $\quad \Delta\lambda = -\frac{2\pi c}{\omega_0^2}\Delta\omega$, where $\Delta\omega = \Gamma$

$$\Rightarrow \Delta\lambda = \frac{2\pi c}{\omega_0^2}\frac{2}{3}\frac{e^2}{c^3 m}\omega_0^2 = \frac{4\pi}{3}\frac{e^2}{mc^2} = 1.2\times10^{-12}\ \text{cm}$$
$$= 1.2\times10^{-4}\ \text{Å}$$

The classical line width is a constant in wavelength, independent of $\lambda_0$. It's also called "natural line width".

## Driven (Forced) Harmonic Oscillator with Damping

is a model for line absorption (or scattering)

$$m\ddot{x} + kx + m\Gamma\dot{x} = eE_0 e^{i\omega t}$$

(forcing term)

Assume a solution $x = Ae^{i\omega t} = |A|e^{i(\omega t+\delta)}$
where $A$ is a complex amplitude.

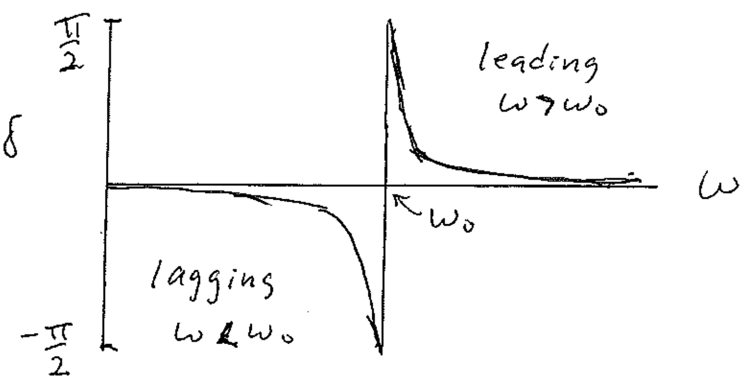
Solution is $\quad -A\omega^2 + A\omega_0^2 + Ai\omega\Gamma = eE_0/m$

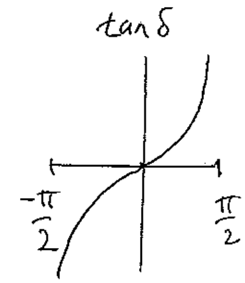
$$A = \frac{-(eE_0/m)}{(\omega^2-\omega_0^2-i\Gamma\omega)} = \frac{-eE_0}{m}\,\frac{(\omega^2-\omega_0^2)+i\Gamma\omega}{(\omega^2-\omega_0^2)^2+\Gamma^2\omega^2}$$

The scattered wave is out of phase with the incident wave.

Since $A = |A|e^{i\delta} = |A|(\cos\delta + i\sin\delta)$

$\tan\delta = \dfrac{\Gamma\omega}{\omega^2-\omega_0^2}$ $\quad$ where $\delta$ is the phase shift




The time-averaged power radiated is $\langle P\rangle = \frac{e^2}{3c^3}|\ddot{x}|^2$
where $|\ddot{x}|^2 = \omega^4|A|^2$

$$\boxed{\langle P\rangle = \frac{e^2}{3c^3}\,\frac{e^2E_0^2}{m^2}\,\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\Gamma^2\omega^2}} \qquad (2)$$

Consider three limiting cases of equation (2)

1. $\omega \gg \omega_0$ This is equivalent to scattering from a free electron, which is Thomson scattering because there is no restoring force ($k=0$).

$$\langle P \rangle \approx \frac{e^4 E_0^2}{3 m^2 c^3}$$

The incident flux is $\langle S \rangle = \frac{c E_0^2}{8\pi}$

The scattering cross section is $\sigma_T \equiv \frac{\langle P \rangle}{\langle S \rangle} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$

2. $\omega \ll \omega_0$ This is the low-frequency regime of Rayleigh scattering, with cross-section $\sigma \propto \frac{1}{\lambda^4}$

$$\langle P \rangle \approx \frac{e^4 E_0^2}{3 m^2 c^3}\left(\frac{\omega}{\omega_0}\right)^4$$

$$\sigma(\omega) = \sigma_T \left(\frac{\omega^4}{\omega_0^4}\right)$$

3. $\omega \approx \omega_0$ This is resonance scattering

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \Gamma^2\omega^2}$$

Let $\omega^2-\omega_0^2 = (\omega+\omega_0)(\omega-\omega_0) \approx 2\omega_0(\omega-\omega_0)$, $\omega \approx \omega_0$

Then $\sigma(\omega) \approx \sigma_T \frac{\omega_0^4}{4\omega_0^2(\omega-\omega_0)^2 + \Gamma^2\omega_0^2} = \sigma_T \frac{(\omega_0/2)^2}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$

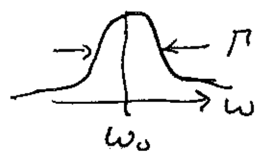
But $\Gamma = \frac{2e^2\omega_0^2}{3mc^3}$ from radiation reaction force.

It's easy to show that, with $\omega = 2\pi\nu$

$$\boxed{\sigma(\omega) = \frac{2\pi^2 e^2}{mc}\frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}}$$

$$\boxed{\sigma(\nu) = \frac{\pi e^2}{mc}\frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}}$$

## Natural (damping) line profile




Lorentzian $\quad \phi(\nu) = \dfrac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} \qquad \int \phi(\nu)\, d\nu = 1$

$$\sigma(\nu) = \frac{\pi e^2}{m_e c} \phi(\nu) f_{12} \qquad \sigma_{12} = \int \sigma(\nu)\, d\nu = \frac{\pi e^2}{m_e c} f_{12}$$

[cm²] [s⁻¹]

$$A_{21} = \Gamma$$

$$B_{12} \frac{h\nu_{12}}{4\pi} = \frac{\pi e^2}{m_e c} f_{12} \qquad \left( \frac{\pi e^2}{m_e c} = 0.0265 \ \frac{cm^2}{s} \right)$$

Check units, $B_{12} J_\nu$ has units of $A_{21}$ [$s^{-1}$]

$$B_{12} \left[ \frac{cm^2\, s\, Hz\, sr}{erg} \ \frac{1}{s} \right] \frac{h\nu_{12}}{4\pi} \left[ \frac{erg}{sr} \right] = \frac{\pi e^2}{m_e c} f_{12} \left[ \frac{cm^2}{s} \right]$$

## Classical radiation reaction

$$\Gamma = \frac{2 e^2 \omega_0^2}{3 m_e c^3} = \frac{8\pi^2 e^2}{3 m_e c \lambda_0^2} = 1.5 \times 10^9 \ s^{-1} \text{ for } \lambda = 1215 \text{ Å}$$

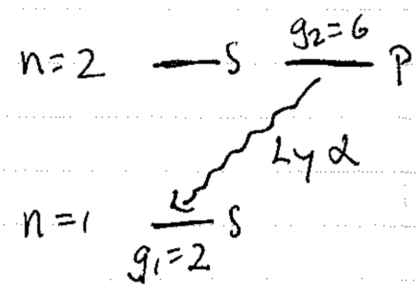
## Quantum oscillator strength $f_{12}$

$f_{12} = 0.4162$ for Ly α

$$A_{21} = \frac{2h\nu_{12}^3}{c^2} B_{21} = \frac{2h\nu_{12}^3}{c^2} \frac{g_1}{g_2} B_{12} = \frac{8\pi^2 e^2 \nu_{12}^2}{m_e c^3} \frac{g_1 f_{12}}{g_2}$$

$$A_{21} = \frac{8\pi^2 e^2}{m_e c \lambda_{12}^2} \frac{g_1}{g_2} f_{12} = 3 \times 1.5 \times 10^9 \times \frac{2}{6} \times 0.4162$$

$$A_{21} = 6.26 \times 10^8 \ s^{-1}$$



## Quantum meaning of $\Gamma_j$ for a level vs. $\Gamma$ for a line

If $\Gamma_j^{-1}$ is the mean lifetime of the upper level $j$, the probability of the electron being in level $j$ is

$$P(t) = \psi_j \psi_j^* e^{-\Gamma_j t}$$

where $\psi_j(\vec{r},t) = u_j(\vec{r})\, e^{-iE_j t/\hbar}$ is the wave function of level $j$.

If there are more than one level $i < j$, then

$$\Gamma_j = \sum_{i<j} A_{ji}$$

If the lower level of a transition also has a finite lifetime, then $\Gamma$ of the transition is

$$\Gamma = \Gamma_j + \Gamma_i$$

A <u>resonance line</u> is a transition to or from the ground state, which has $\Gamma_i = 0$, so it is narrower than non-resonance lines, because it has a smaller $\Gamma$.

<u>Cross section at line center</u> $(\nu = \nu_0)$

$$\sigma(\nu) = \frac{\pi e^2}{m_e c} \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} f_{12} = \frac{4\pi e^2 f_{12}}{m_e c \Gamma}$$

$$\sigma(\nu_0) = \frac{4\pi e^2}{m_e c} \frac{f_{12}}{A_{21}} \quad (= 7\times10^{-11}\ \text{cm}^2 \text{ for Ly}\alpha)$$

<u>Doppler Broadening</u>

What if $\frac{\Delta\nu}{\nu_0} = \frac{v}{c} \geq \frac{\Gamma}{\omega_0} \approx \frac{10^9}{10^{16}} \approx 10^{-7}$, or $v > 10\ \frac{m}{s}$