

Maxwell's Equations (in Gaussian cgs units)

$$\nabla \cdot \vec{E} = 4\pi\rho$$

Gauss's law

$$\nabla \cdot \vec{B} = 0$$

no magnetic monopole

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

Faraday's law

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Ampere's law

$\rho \left[ \frac{esu}{cm^3} \right]$  charge density



$$\vec{J} = \vec{I}/A$$

$$\vec{J} = \rho \vec{v}$$

$\vec{J}$  = current density  $\left[ \frac{esu}{cm^2 s} \right]$

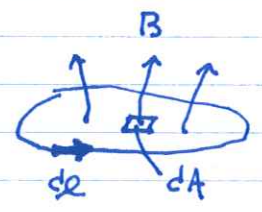
Integral form of Maxwell's Equations

$$\oint_S \vec{E} \cdot d\vec{A} = 4\pi \int_V \rho dV = 4\pi Q_{enc}$$

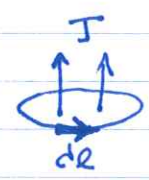


$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

$$\int (\nabla \times \vec{E}) \cdot d\vec{A} = \oint_L \vec{E} \cdot d\vec{r} = -\frac{1}{c} \frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{A}$$



$$\begin{aligned} \oint_L \vec{B} \cdot d\vec{r} &= \frac{4\pi}{c} \int \vec{J} \cdot d\vec{A} + \frac{1}{c} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A} \\ &= \frac{4\pi}{c} I_{enc} + \frac{1}{c} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A} \end{aligned}$$

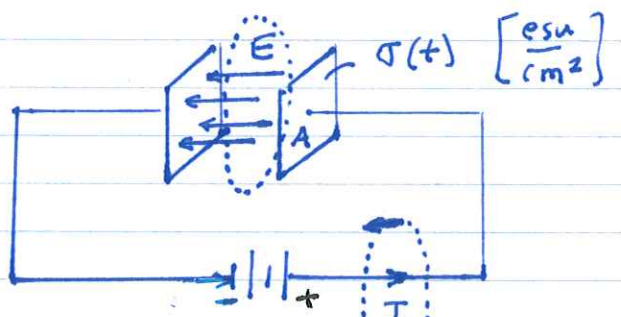


Displacement Current

If the plates are close together

$$\vec{E} \cdot \vec{A} = 4\pi \sigma A$$

$$\frac{\partial E}{\partial t} = 4\pi \frac{\partial \sigma}{\partial t} = 4\pi \frac{I}{A}$$



$$\frac{1}{c} \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} = \frac{4\pi}{c} I_d \leftarrow \text{Maxwell called this displacement current}$$

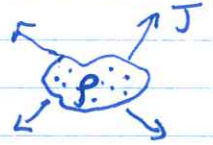
If not for this term we would have only  $\nabla \times \vec{B} = 4\pi \vec{J}/c$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{B}) = \frac{4\pi}{c} \nabla \cdot \vec{J} = 0$$

But with the displacement current term:

$$4\pi \nabla \cdot \vec{J} = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) \quad \text{combined with} \quad \nabla \cdot \vec{E} = 4\pi \rho \quad \text{gives}$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad , \quad \text{the continuity equation}$$



We also need the Lorentz Force Law

$$\vec{F} = q (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \quad \text{or} \quad \vec{f} = \rho \vec{E} + \frac{\vec{J}}{c} \times \vec{B} \quad \left[ \frac{\text{force}}{\text{vol}} \right]$$

Dielectrics (if electrons are bound to atoms and don't conduct)

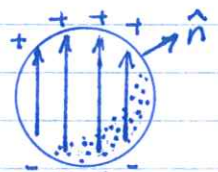
$$\rho = \rho_b + \rho_f \quad \text{charge density has both bound and free parts}$$

Bound charge creates a Polarization  $\vec{P}$  defined as

$$\nabla \cdot \vec{P} = -\rho_b \quad \vec{E} \rightarrow \left( \text{cylinder} \right) \rightarrow \vec{P} \rightarrow \vec{E}$$

Polarization  $\vec{P}$  is dipole moment per unit volume

$$\text{Dipole moment } \vec{d} = \sum_i q_i \vec{r}_i \quad \text{or} \quad \int \rho \vec{r} dV$$



$$\nabla \cdot \vec{E} = 4\pi (\rho_b + \rho_f) = 4\pi (-\nabla \cdot \vec{P} + \rho_f)$$

$$\nabla_b = \vec{P} \cdot \hat{n}$$

$$\nabla \cdot (\vec{E} + 4\pi \vec{P}) = 4\pi \rho_f$$

$$\vec{D} = \vec{E} + 4\pi \vec{P} \quad \vec{D} = \text{Displacement}$$

Gauss's law in matter

$$\nabla \cdot \vec{D} = 4\pi \rho_f$$

In a linear medium  $\vec{P} = \chi_e \vec{E}$ , where  $\chi_e$  is the electric susceptibility.

$$\vec{D} = \vec{E} + 4\pi \chi_e \vec{E} = \epsilon \vec{E} \quad \epsilon = \text{Dielectric constant}$$

$$\epsilon = 1 + 4\pi \chi_e$$

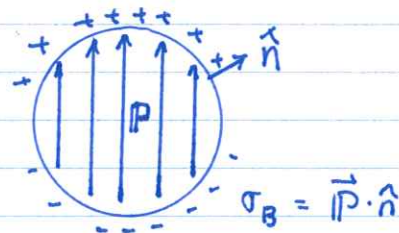
	$\epsilon$
air	1.00059
glass	4-7
water	80.4

## Rayleigh Scattering - derivation of differential cross section

A dielectric sphere of radius  $r$  that is much smaller than the wavelength of incident light  $\lambda$  will be polarized uniformly and in phase with the applied electric field of the plane wave. The sphere will have an oscillating bound surface charge density and an induced dipole moment.

$$\sigma_b = \vec{P} \cdot \hat{n} = P \cos \theta$$

$$\vec{d} = \frac{4\pi r^3}{3} \vec{P}$$



An individual molecule will have an induced dipole moment  $\vec{p}$  caused by the local electric field from all the other molecules and sources of field around it,  $\vec{E}_{\text{others}}$ .

$$\vec{p} = \alpha \vec{E}_{\text{others}}$$

where  $\alpha$  is the atomic polarizability

The total electric field  $\vec{E}$  is due to the molecule itself  $\vec{E}_{\text{self}}$  plus  $\vec{E}_{\text{others}}$ .

$$\vec{E} = \vec{E}_{\text{self}} + \vec{E}_{\text{others}}$$

Let the number density of molecules be  $n \approx \frac{3}{4\pi a^3}$  where  $a$  is the radius of a molecule.

Since the polarization  $\vec{P}$  is the dipole moment per unit volume,

$$\vec{P} = n \alpha \vec{E}_{\text{others}}$$

The average dipole field inside a molecule is not calculated here, but it is

$$\vec{E}_{\text{self}} = -\frac{\vec{p}}{a^3} = -\alpha \frac{\vec{E}_{\text{others}}}{a^3}$$

Therefore,

$$\vec{E} = \left(1 - \frac{\alpha}{a^3}\right) \vec{E}_{\text{others}}$$

$$\text{and } \vec{P} = n\alpha \vec{E}_{\text{others}} = \frac{n\alpha}{(1 - \frac{4\pi}{3}n\alpha)} \vec{E}$$

On page 52 we defined the electric susceptibility  $\chi_e$  of a linear dielectric as  $\vec{P} = \chi_e \vec{E}$ , and the dielectric constant  $\epsilon = 1 + 4\pi\chi_e$ . Therefore

$$\chi_e = \frac{n\alpha}{(1 - \frac{4\pi}{3}n\alpha)} = \frac{\epsilon - 1}{4\pi}$$

Solving this last equation for  $\alpha$  as a function of  $n$  and  $\epsilon$  gives

$$\alpha = \frac{3}{4\pi n} \left( \frac{\epsilon - 1}{\epsilon + 2} \right) = a^3 \left( \frac{\epsilon - 1}{\epsilon + 2} \right)$$

Thus we have related the microscopic quantity, atomic polarizability, to the macroscopic quantity, dielectric constant. Since the dipole moment of the sphere is its polarization times its volume

$$\vec{d} = \frac{4\pi r^3}{3} \vec{P} = \frac{4\pi r^3}{3} n\alpha \vec{E}_{\text{others}} = r^3 \left( \frac{\epsilon - 1}{\epsilon + 2} \right) \vec{E}(t)$$

where  $\vec{E}_{\text{others}} = \vec{E}(t)$ , the electric field of the incident radiation. The power radiated per unit solid angle is, according to the usual dipole formula (page 41),

$$\frac{dP}{d\Omega} = \frac{\ddot{d}^2 \sin^2\theta}{4\pi c^3}$$

$$\text{Let } \vec{E}(t) = E_0 \cos(\omega t) \quad \text{where } \omega = \frac{2\pi c}{\lambda}$$

$$\text{Then } \ddot{d}(t) = -\omega^2 d(t) = -\frac{4\pi^2 c^2 r^3}{\lambda^2} \left( \frac{\epsilon - 1}{\epsilon + 2} \right) E(t)$$

and

$$\frac{dP}{d\Omega} = \frac{4\pi^3 c r^6}{\lambda^4} \left( \frac{\epsilon - 1}{\epsilon + 2} \right)^2 \sin^2\theta E^2(t)$$

The incident flux is  $\vec{S} = \frac{c}{4\pi} E_0^2 \cos^2(\omega t)$

The differential cross section for Rayleigh scattering is the time average ratio of scattered power to incident flux:

$$\frac{d\sigma_R}{d\Omega} = \left\langle \frac{dP}{d\Omega} \right\rangle / \langle S \rangle = \frac{2\pi^3 c r^6}{\lambda^4} \left( \frac{\epsilon-1}{\epsilon+2} \right)^2 \sin^2 \theta / \frac{c}{8\pi}$$

$$\boxed{\frac{d\sigma_R}{d\Omega} = \frac{16\pi^4}{\lambda^4} r^6 \left( \frac{\epsilon-1}{\epsilon+2} \right)^2 \sin^2 \theta}$$

The total cross section for Rayleigh scattering is

$$\sigma_R = \int \frac{d\sigma_R}{d\Omega} d\Omega = \frac{16\pi^4 r^6}{\lambda^4} \left( \frac{\epsilon-1}{\epsilon+2} \right)^2 \int_{-1}^1 \underbrace{(1-\mu^2)}_{4/3} d\mu \int_0^{2\pi} d\phi$$

$$\boxed{\sigma_R = \frac{128\pi^5}{3\lambda^4} r^6 \left( \frac{\epsilon-1}{\epsilon+2} \right)^2}$$

Aside: in the case of a sphere of dielectric constant  $\epsilon_2$  embedded in a medium of dielectric constant  $\epsilon_1$ , the more general result is:

$$\alpha = a^3 \left( \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + 2\epsilon_1} \right)$$

$$\sigma_R = \frac{128\pi^5}{3\lambda^4} r^6 \left( \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + 2\epsilon_1} \right)^2, \text{ etc ...}$$

Compare the Rayleigh cross section to the Thomson cross section,

$$\sigma_T = \frac{8\pi}{3} r_0^2 = \frac{8\pi}{3} \left( \frac{e^2}{m_e c^2} \right)^2 \quad \text{where } r_0 = 2.82 \times 10^{-13} \text{ cm}$$

Therefore

$$\frac{\sigma_R}{\sigma_T} = \left( \frac{2\pi r}{\lambda} \right)^4 \left( \frac{r}{r_0} \right)^2 \left( \frac{\epsilon-1}{\epsilon+2} \right)^2$$

They have the same angular dependence and polarizing effect.

Measurement of Avogadro's number from Rayleigh scattering, which is the cause of atmospheric opacity at visible wave lengths. The optical depth of the atmosphere is

$$\tau = \int_0^{\infty} n \sigma_R dz = 0.23 \text{ at } \lambda = 4500 \text{ \AA}$$

From atmospheric pressure we know that

$$\int_0^{\infty} p dz = 1033 \text{ g cm}^{-2}$$

If approximated as pure N<sub>2</sub> with 28 g mol<sup>-1</sup>

$$N = \int_0^{\infty} n dz = \int \frac{N_A}{28} p dz = 2.2 \times 10^{25} \text{ cm}^{-2}$$

where  $N_A = 6.02 \times 10^{23} \text{ mol}^{-1}$  (Avogadro's number)

At sea level  $n = 2.6 \times 10^{19} \text{ cm}^{-3}$  (Loschmidt's number)

These numbers are related through the exponential scale height of the atmosphere,  $h_s$ , where  $N = nh_s$ .

For the radius of a scatterer, use the average volume occupied by a molecule at sea level

$$\frac{4\pi r^3}{3} = \frac{1}{n} \Rightarrow r^3 = \left( \frac{3}{4\pi n} \right)^2$$

The dielectric constant of air is  $\epsilon = 1.00059$ , so  $\epsilon + 2 \approx 3$ . Substituting  $r^3$  and  $\epsilon + 2 = 3$  into the Rayleigh cross section gives

$$\sigma_R = \frac{8\pi^3}{3\lambda^4} \frac{(\epsilon - 1)^2}{n^2} = 1.04 \times 10^{-26} \text{ cm}^2$$

$$\text{Then } \tau = \int n \sigma_R dz = 2.2 \times 10^{25} \times 1.04 \times 10^{-26} = 0.23$$

Note that in this analysis  $r = 2 \times 10^{-7}$  cm which is much less than the wavelength of light,  $4.5 \times 10^{-5}$  cm, as required for Rayleigh scattering to apply. For this choice of radius, the Rayleigh cross section doesn't depend on density since the dielectric constant also is a function of density. Specifically

$$\epsilon - 1 = 4\pi \chi_e = \frac{4\pi n \alpha}{1 - \frac{4\pi}{3} n \alpha} \approx n \quad (\text{since } n \alpha \ll 1)$$

The Rayleigh cross section doesn't depend on  $r$  either, because it is the fluctuations in density that cause the scattering. In any volume  $V$  containing  $N$  molecules on average, the mean square fluctuation in the number of particles is  $(\Delta N)^2$ , where

$$\frac{(\Delta N)^2}{N} = -\frac{P}{V} \left( \frac{dV}{dP} \right)_T = 1 \quad (\text{for an ideal gas}^*)$$

$$\frac{1}{V} \left( \frac{dV}{dP} \right)_T \text{ is the "isothermal compressibility"}$$

$$= \frac{1}{V} \left( \frac{-NRT}{P^2} \right) = -\frac{1}{P} \quad \text{for an ideal gas}$$

$$\Delta N = \sqrt{N} \quad \text{where } N = \frac{4\pi}{3} r^3 n$$

Think of a volume containing  $\Delta N$  more particles than average. It behaves like a sphere of dielectric constant  $\epsilon_2$  embedded in a medium of dielectric constant  $\epsilon_1$ . Then

$$\sigma_R \propto r^6 \left( \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + 2\epsilon_1} \right)^2 \quad \text{where } \epsilon_2 - 1 \approx n_2$$

$$\epsilon_1 - 1 \approx n_1$$

\*  $PV = \mathcal{N}RT$   
or  $P = nkT$

$$R = N_A k$$

$$n = \mathcal{N} \frac{N_A}{V}$$

$$\mathcal{N} = \# \text{ of moles}$$

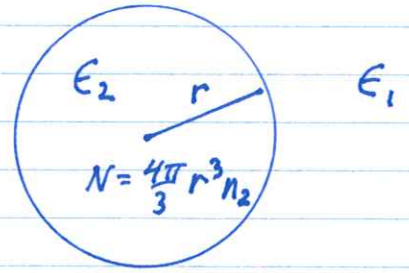
$$N_A = \text{Avogadro's } \#$$

$$\Delta N = \sqrt{N} \propto r^{3/2} n^{1/2}$$

$$E_2 - E_1 \propto \frac{\Delta N}{r^3} \propto \frac{n^{1/2}}{r^{3/2}}$$

$$\sigma_R \propto r^6 (E_2 - E_1)^2$$

$$\propto r^6 \frac{n}{r^3} \propto r^3 n \propto N$$



For a given density,  $\sigma_R$  doesn't depend on radius, just on the total number of particles  $N$ . So every molecule contributes equally to the optical depth, no matter where it is in density. It is the fluctuations that are responsible for scattering.