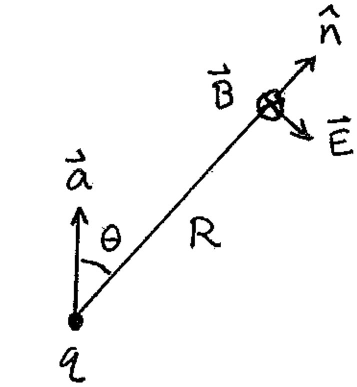
Radiation from accelerated charges



$\vec{E} = \frac{q}{c^2 R} \hat{n} \times (\hat{n} \times \vec{a})$ non-relativistic

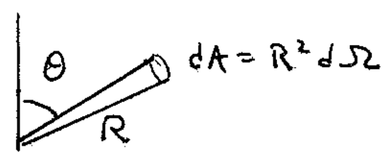
$\vec{B} = \hat{n} \times \vec{E}$

$|\vec{E}| = |\vec{B}| = \frac{q}{c^2 R} a \sin\theta$

Poynting vector $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$ gives the

energy flux $S = \frac{q^2}{4\pi c^3 R^2} a^2 \sin^2\theta$ $\left[\frac{\text{erg}}{\text{cm}^2 \text{ s}}\right]$

Total power $P = \int S \, dA$ $\left[\frac{\text{erg}}{\text{s}}\right]$



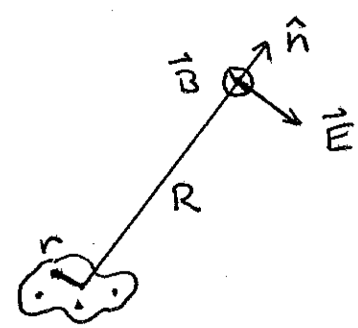
$$P = \frac{q^2 a^2}{4\pi c^3} \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{q^2 a^2}{4\pi c^3} \underbrace{\int_{-1}^{1} (1-\mu^2) \, d\mu}_{4/3} \cdot 2\pi$$

$$\boxed{P = \frac{2}{3} \frac{q^2}{c^3} a^2}$$ Larmor's Formula

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\theta}$$ Power per solid angle

System of Charges



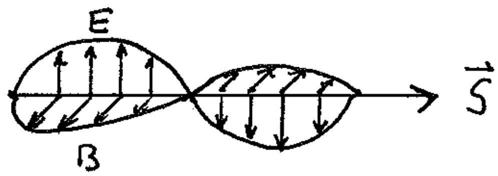
$\vec{d} = \sum_i q_i \vec{r}_i$ Dipole moment

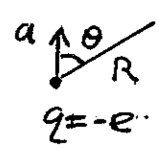
$\ddot{\vec{d}} = \sum_i q_i \ddot{\vec{r}}_i$

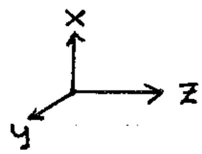
$\vec{E} = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R}$ If non-relativistic and $R \gg r$

Thomson Scattering (if $h\nu \ll m_e c^2$)

The cross section for Thomson scattering can be calculated using classical electrodynamics.







$$\vec{E} = E_0\, e^{i(kz-\omega t)}\,\hat{x}$$
$$\vec{B} = B_0\, e^{i(kz-\omega t)}\,\hat{y}$$
incident wave

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$
force on charge (neglect $\frac{v}{c}$ term)

$$m_e a = -e E_0 \cos(\omega t)$$
in x-direction only

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2 \sin^2\theta \cos^2(\omega t)}{4\pi c^3 m_e^2}$$
Power scattered per solid angle

Let $\langle \cos^2(\omega t)\rangle = \frac{1}{2}$ in a time average

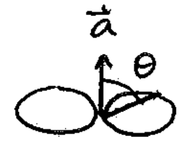
$$\left\langle \frac{dP}{d\Omega}\right\rangle = \frac{e^4 E_0^2}{8\pi m_e^2 c^3}\sin^2\theta \qquad \left[\frac{\text{erg}}{\text{ster s}}\right]$$

$\left[\frac{\text{erg}}{\text{cm}^2\,\text{s}}\right]$ $\quad \langle S\rangle = \frac{c}{8\pi}E_0 B_0 = \frac{c}{8\pi}E_0^2$ time average incident flux

Differential cross section $\quad \frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega\rangle}{\langle S\rangle} \quad \left[\frac{\text{cm}^2}{\text{ster}}\right]$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4}\sin^2\theta = r_0^2 \sin^2\theta$$

where $r_0 = \frac{e^2}{m_e c^2} = 2.82\times 10^{-13}$ cm, the classical electron radius



The angular distribution of scattered power is a <u>torus</u> around the axis defined by the acceleration vector.

The total Thomson cross section is

$$\sigma_T = \int \frac{d\sigma}{d\Omega}\, d\Omega = r_0^2 \int_0^{\pi} \sin^2\theta \, \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\sigma_T = r_0^2 \int_{-1}^{1} (1-\mu^2)\, d\mu \cdot 2\pi = \frac{8\pi}{3} r_0^2$$

$$\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$$

Next, we will talk about polarization, and why Thomson scattering polarizes light. Here is an example:

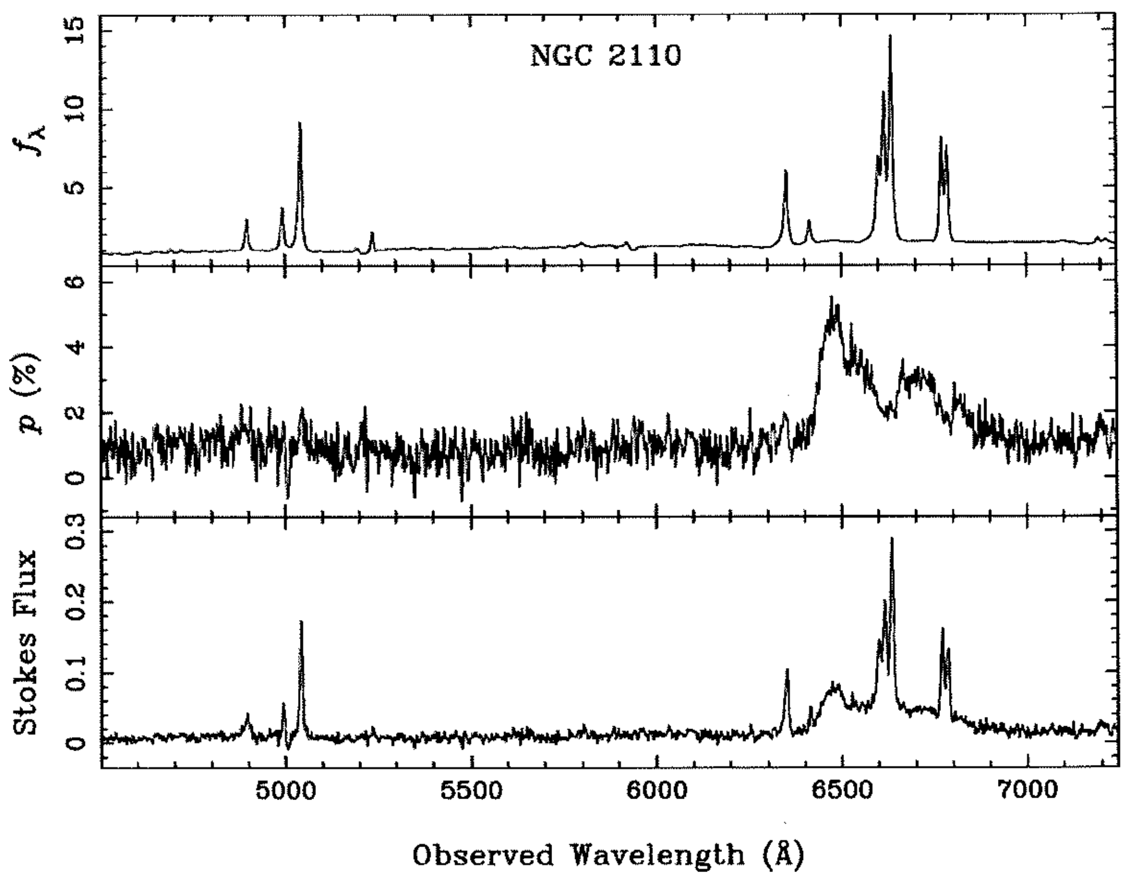
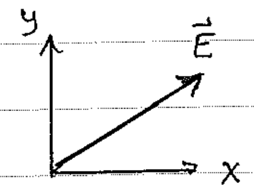


FIG. 1.—Spectropolarimetry of NGC 2110. *Top*: Total flux, in units of 10^{-15} ergs cm^{-2} s^{-1} Å^{-1}. *Middle*: Degree of linear polarization, given as the rotated Stokes parameter. *Bottom*: Polarized flux, or "Stokes flux," which is the product of the total flux and rotated Stokes parameter.

Polarization and Stokes Parameters

Consider a monochromatic wave propagating in the z-direction, which is toward you, the observer. Its electric field components are in the x- and y-directions.

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} = (E_1 \hat{x} + E_2 \hat{y}) e^{i(kz-\omega t)}$$



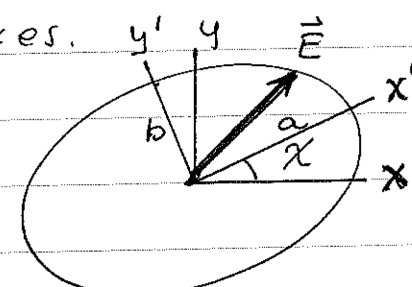
In general, E_1 and E_2 are complex numbers to account for possible difference in phase.
$\begin{cases} E_1 = \mathcal{E}_1 e^{i\phi_1} \\ E_2 = \mathcal{E}_2 e^{i\phi_2} \end{cases}$

Physical field are real. Evaluate them at $z=0$.

$$\vec{E}(0,t) = \mathcal{E}_1 \cos(\omega t - \phi_1)\hat{x} + \mathcal{E}_2 \cos(\omega t - \phi_2)\hat{y}$$

$$E_x = \mathcal{E}_1 (\cos\omega t \cos\phi_1 + \sin\omega t \sin\phi_1) \qquad (1)$$
$$E_y = \mathcal{E}_2 (\cos\omega t \cos\phi_2 + \sin\omega t \sin\phi_2) \qquad (2)$$

Consider an ellipse whose principal axes are tilted at angle χ with respect to x- and y-axes. The parametric equations of the ellipse are:



$$E'_{x'} = \mathcal{E}_0 \cos\beta \cos\omega t$$
$$E'_{y'} = -\mathcal{E}_0 \sin\beta \sin\omega t$$

$$\boxed{\begin{gathered} \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1 \\ a^2 + b^2 = \mathcal{E}_0^2 \\ a = \mathcal{E}_0 \cos\beta \\ b = \mathcal{E}_0 \sin\beta \end{gathered}}$$

Use the rotation matrix to transform from prime to unprime coordinates

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} E'_{x'} \\ E'_{y'} \end{pmatrix}$$

$$E_x = \mathcal{E}_0 \cos\beta \cos\omega t \cos\chi + \mathcal{E}_0 \sin\beta \sin\omega t \sin\chi$$
$$E_y = \mathcal{E}_0 \cos\beta \cos\omega t \sin\chi - \mathcal{E}_0 \sin\beta \sin\omega t \cos\chi$$

These equations are identical to equations (1) and (2) above if we make the following identifications

$$\boxed{\begin{aligned} \mathcal{E}_1 \cos\phi_1 &= \mathcal{E}_0 \cos\beta \cos\chi \\ \mathcal{E}_1 \sin\phi_1 &= \mathcal{E}_0 \sin\beta \sin\chi \\ \mathcal{E}_2 \cos\phi_2 &= \mathcal{E}_0 \cos\beta \sin\chi \\ \mathcal{E}_2 \sin\phi_2 &= -\mathcal{E}_0 \sin\beta \cos\chi \end{aligned}} \qquad (3)$$

If $\beta = 0$ $\quad \mathcal{E}_{x'}' = \mathcal{E}_0$, $\mathcal{E}_{y'}' = 0$ $\quad$ linear polarization x'

$\beta = \pm\frac{\pi}{2}$ $\quad \mathcal{E}_{x'}' = 0$, $\mathcal{E}_{y'}' = \mathcal{E}_0$ $\quad$ linear polarization y'

$\beta = \pm\frac{\pi}{4}$ $\quad \phi_1 - \phi_2 = \pm\pi/2$ $\quad$ circular polarization

Polarization can be specified completely by $\mathcal{E}_0, \beta, \chi$ instead of $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$. Equations (3) are four equations for three unknowns. First square them:

$$\mathcal{E}_1^2 = \mathcal{E}_0^2 (\cos^2\beta \cos^2\chi + \sin^2\beta \sin^2\chi)$$
$$\mathcal{E}_2^2 = \mathcal{E}_0^2 (\cos^2\beta \sin^2\chi + \sin^2\beta \cos^2\chi)$$
$$\mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 [\cos^2\beta (\cos^2\chi - \sin^2\chi) - \sin^2\beta (\cos^2\chi - \sin^2\chi)]$$

using trig identity $\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$$\mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 (\cos 2\beta \cos 2\chi)$$

Define $$\boxed{\begin{aligned} I &\equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \\ Q &\equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \end{aligned}}$$

Now multiply equations (3) together in pairs:

$$\mathcal{E}_1 \mathcal{E}_2 \cos\phi_1 \cos\phi_2 = \mathcal{E}_0^2 \cos^2\beta \cos\chi \sin\chi$$
$$\mathcal{E}_1 \mathcal{E}_2 \sin\phi_1 \sin\phi_2 = -\mathcal{E}_0^2 \sin^2\beta \cos\chi \sin\chi$$
$$\mathcal{E}_1 \mathcal{E}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) = \mathcal{E}_0^2 \cos\chi \sin\chi (\cos^2\beta - \sin^2\beta)$$
$$\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \tfrac{1}{2} \sin 2\chi \cos 2\beta$$

Define $\boxed{U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\chi \cos 2\beta}$

Multiply another combination of equations (3) in pairs

$$\mathcal{E}_1 \cos\phi_1 \, \mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0^2 \cos\beta \sin\beta \cos^2\chi$$
$$\mathcal{E}_1 \sin\phi_1 \, \mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0^2 \sin\beta \cos\beta \sin^2\chi$$
$$\mathcal{E}_1 \mathcal{E}_2 (\sin\phi_1 \cos\phi_2 - \cos\phi_1 \sin\phi_2) = \mathcal{E}_0^2 \cos\beta \sin\beta$$
$$\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \tfrac{1}{2} \sin 2\beta$$

Define $\boxed{V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta}$

Stokes parameters are I, Q, U, V

- I is the total intensity
- V is the circularly polarized intensity
  - $V > 0$ is right handed
  - $V < 0$ is left handed
- Q and U measure the orientation of the ellipse

$$\frac{U}{Q} = \tan 2\chi$$

$U = 0$ when $\chi = 0$

$Q = 0$ when $\chi = \frac{\pi}{4}$

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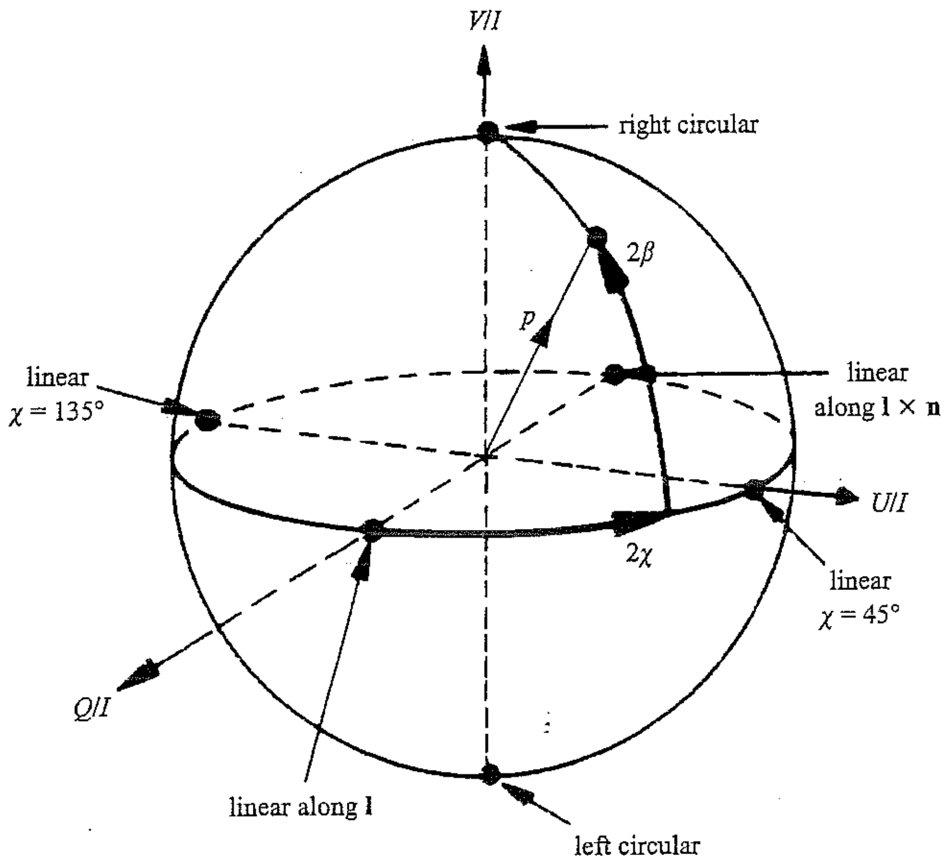
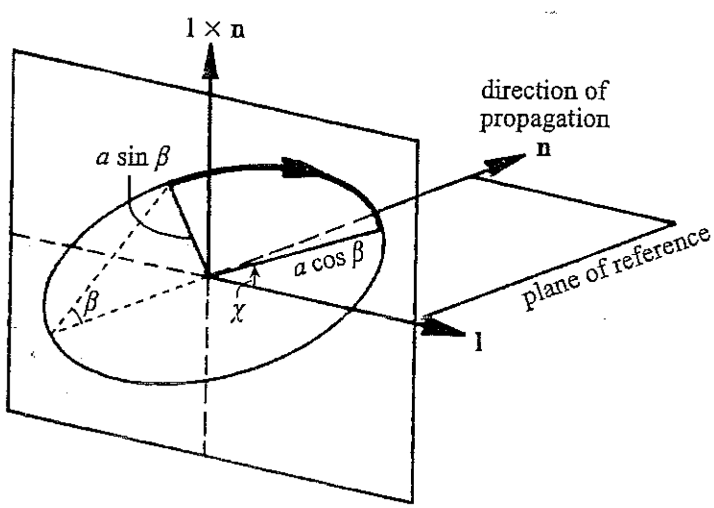


Polarization ellipse



Stokes parameters

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} a^2 \\ a^2 \cos 2\beta \cos 2\chi \\ a^2 \cos 2\beta \sin 2\chi \\ a^2 \sin 2\beta \end{pmatrix}$$

$$\chi = \text{polarization angle}$$
$$\tan\beta = \text{axial ratio of ellipse}$$

$V =$	0	-1	0	$+1$	0
$\beta =$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$+\frac{\pi}{4}$	$+\frac{\pi}{2}$
	y'	LC	x'	RC	y'

You can show that the following operational definitions of the Stokes parameters hold:

Q is the difference between intensities transmitted by a linear polarizer oriented along the x-axis and one oriented along the y-axis

U is the difference between intensities transmitted by a linear polarizer oriented at $45°$ and one oriented at $135°$.

V is the difference between intensity transmitted by a right circular polarizer and a left circular polarizer

Polarizers can be represented as unit vectors:

Linear along x	$\hat{x}$
Linear along y	$\hat{y}$
Linear along $45°$	$(\hat{x} + \hat{y})/\sqrt{2}$
Linear along $135°$	$(-\hat{x} + \hat{y})/\sqrt{2}$
Right circular	$(\hat{x} + i\hat{y})/\sqrt{2}$
Left circular	$(\hat{x} - i\hat{y})/\sqrt{2}$

For example, intensity transmitted by an $\hat{x}$ polarizer is

$$|\hat{x} \cdot \vec{E}|^2$$

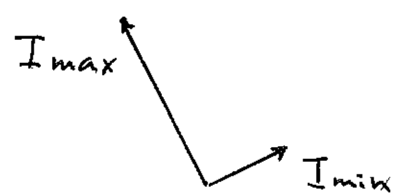
where $\vec{E} = \mathcal{E}_1 e^{i(\phi_1 - \omega t)} \hat{x} + \mathcal{E}_2 e^{i(\phi_2 - \omega t)} \hat{y}$.

Therefore $|\hat{x} \cdot E|^2 = \mathcal{E}_1^2$

The degree (or percentage) of polarization is:

$$\Pi = \frac{I_{pol}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$

<u>Example: Partial Linear Polarization</u>



$$I_{max} = I_{pol} + \frac{1}{2} I_{unpol}$$

$$I_{min} = \frac{1}{2} I_{unpol}$$

where $I_{pol} = \sqrt{Q^2 + U^2}$

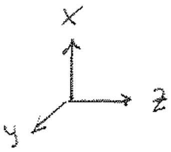
$$I_{unpol} = 1 - \sqrt{Q^2 + U^2}$$

The degree of polarization is

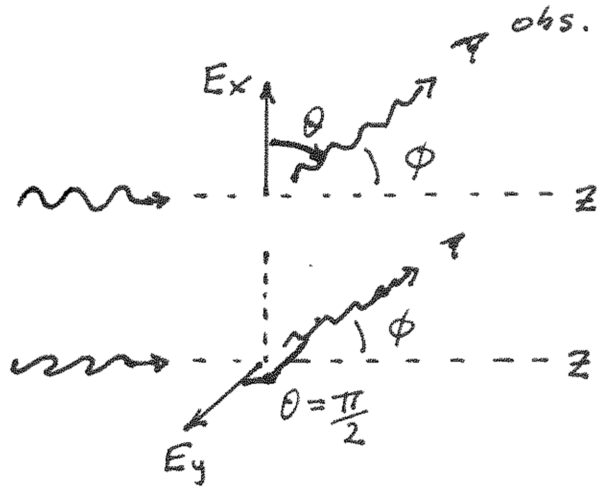
$$\Pi = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{I_{pol}}{I_{pol} + I_{unpol}}$$

If light is 100% polarized $I^2 = Q^2 + U^2 + V^2$, otherwise $Q^2 + U^2 + V^2 < I^2$, where $I = I_{pol} + I_{unpol}$

Polarization of Thomson Scattered Radiation



An unpolarized wave can be considered as a superposition of equally intense linearly polarized waves, polarized in the x- and y-directions.



$$\left(\frac{d\sigma}{d\Omega}\right)_x = r_0^2 \sin^2\theta$$

x- polarization

$$\left(\frac{d\sigma}{d\Omega}\right)_y = r_0^2$$

y - polarization

ϕ is the scattering angle.
The cross section for unpolarized radiation is the average of the values for x- and y-polarization.

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} r_0^2 (1 + \sin^2\theta) = \frac{1}{2} \frac{e^4}{m_e^2 c^4} (1 + \cos^2\phi)$$

It depends only on the scattering angle ϕ, between the photon's incident and scattered directions.

Now consider the polarization of the scattered radiation. The intensities of the scattered components are denoted I_x and I_y, which are proportional to $(d\sigma/d\Omega)_x$ and $(d\sigma/d\Omega)_y$, respectively. The degree of polarization is:

$$\Pi = \frac{I_y - I_x}{I_y + I_x} = \frac{1 - \sin^2\theta}{1 + \sin^2\theta} = \frac{1 - \cos^2\phi}{1 + \cos^2\phi} \qquad \left(\begin{array}{l} \Pi = 1 \text{ when } \phi = 90^\circ \\ = 0 \text{ when } \phi = 0^\circ \end{array}\right)$$

The direction is perpendicular to the plane of the incident and scattered ray.