Waves in a Conducting Plasma

Maxwell's Equations
$$\nabla \cdot E = 4\pi\rho \qquad \nabla \cdot B = 0$$
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla \times B = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

Conductivity σ defined as $J = \sigma E$

so
$$\nabla \times B = \frac{4\pi}{c}\sigma E + \frac{1}{c}\frac{\partial E}{\partial t} \qquad (1)$$

$$\nabla \times (\nabla \times E) = -\frac{1}{c}\nabla \times \frac{\partial B}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) \qquad (2)$$

Vector identity $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E \qquad (3)$

Assume no net charge density. $\rho = 0$, so $\nabla \cdot E = 0$
Combining (1) (2) and (3)

$$-\nabla^2 E = \frac{-1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\sigma E + \frac{1}{c}\frac{\partial E}{\partial t}\right)$$

$$\boxed{\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi\sigma}{c^2}\frac{\partial E}{\partial t}}$$

Wave equation with damping term

also
$$\nabla \times (\nabla \times B) = -\nabla^2 B = \nabla \times \left(\frac{4\pi\sigma E}{c}\right) + \frac{1}{c}\frac{\partial}{\partial t}\nabla \times E$$

$$\boxed{\nabla^2 B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} + \frac{4\pi\sigma}{c^2}\frac{\partial B}{\partial t}}$$

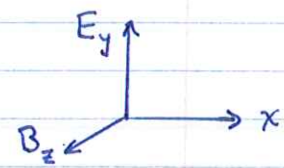
Assume solution of the form $\vec{E}(x,t) = \vec{E}_0 e^{i(kx-\omega t)}$ and substitute into the wave equation. Then we find

$$k^2 = \frac{\omega^2}{c^2} + i\left(\frac{4\pi\sigma\omega}{c^2}\right)$$

<u>a complex wavenumber</u>

A good conductor is defined by $\sigma \gg \omega$ so that the imaginary part of k^2 is larger than the real part.
But in a ionized plasma, σ itself is imaginary, as we show...

Assume an EM wave propagating in the x-direction with E-field in the y-direction. An electron will oscillate with the frequency of the wave, ω, and amplitude y_o.



$$y = y_o e^{-i\omega t}, \qquad \dot{y} = -i\omega y_o e^{-i\omega t}$$

$$\ddot{y} = -\frac{e\vec{E}}{m} = -i\omega\dot{y}, \quad \text{therefore } \dot{y} = \frac{e\vec{E}}{i\omega m}$$

Since current density in an ionized plasma is $J = -nev$ where n is the electron density, and $v = \dot{y}$ in this case,

$$-\frac{\vec{J}}{ne} = \frac{e\vec{E}}{i\omega m} \qquad \text{where } \vec{J} = \sigma\vec{E}, \text{ therefore } \boxed{\sigma = \frac{ine^2}{\omega m}}$$

Since σ is imaginary $\vec{J}$ is $90°$ out of phase with $\vec{E}$ and no net work is done on the charge. Substitute this imaginary conductivity into the wavenumber relation:

$$k^2 = \frac{\omega^2}{c^2} - \frac{4\pi ne^2}{mc^2}$$

Define plasma frequency ω_p as

$$\omega_p^2 = \frac{4\pi ne^2}{m}$$

Then $k = \dfrac{\sqrt{\omega^2 - \omega_p^2}}{c}$

or $\omega^2 = k^2c^2 + \omega_p^2$ This is the dispersion relation in a perfectly conducting plasma, which is dissipationless.

In the case that $\omega < \omega_p$, the wavenumber k is imaginary, and the wave amplitude decreases exponentially

$$E(x,t) = E_o e^{-\sqrt{\omega_p^2 - \omega^2}\, x/c} e^{-i\omega t}$$

with scale length $c/\sqrt{\omega_p^2 - \omega^2}$, also called the "skin depth" beyond which radiation does not penetrate.

Numerically $\omega_p = 5.63 \times 10^4\, n^{1/2}$

In the earth's ionosphere, $n \simeq 10^4$ and frequencies less than 10^6 Hz cannot propagate. In the interstellar medium $n \lesssim 0.1$. Waves of $\omega < \omega_p$ will either be reflected, or absorbed as kinetic energy.

<u>Digression on phase velocity and group velocity</u>

When $\omega > \omega_p$, k is real and the wave propagates with phase velocity $v_{ph} = \frac{\omega}{k}$, which is related to the index of refraction $n_r = \frac{c}{v_{ph}}$. So $n_r = \frac{ck}{\omega} = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$

n_r is less than 1, so phase velocity is $> c$. The wave energy propagates at the group velocity,

$$v_{gr} = \frac{d\omega}{dk} = \frac{d\omega\, 2c\sqrt{\omega^2 - \omega_p^2}}{2\omega\, d\omega} = c\sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$$

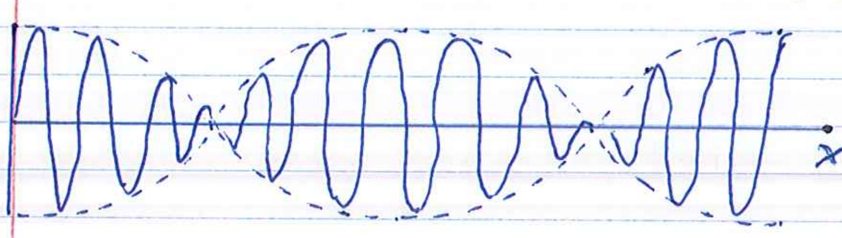
which is $< c$. To understand phase and group velocities, consider a "wave packet" consisting of two frequencies separated by a small $\Delta\omega = \omega_1 - \omega_2$.

$$\left.\begin{aligned} E_1 &= E_0 \sin(k_1 x - \omega_1 t) \\ E_2 &= E_0 \sin(k_2 x - \omega_2 t) \end{aligned}\right\} \text{ Sum these}$$

$$E = 2E_0 \sin\left[\left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\omega_1 + \omega_2}{2}\right)t\right] \cos\left[\left(\frac{k_1 - k_2}{2}\right)x - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right]$$

let $k_1 \approx k_2$, $\omega_1 \approx \omega_2$, $k = \frac{1}{2}(k_1 + k_2)$, $\omega = \frac{1}{2}(\omega_1 + \omega_2)$

then $E = 2E_0 \sin(kx - \omega t) \cos\left(\frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t\right)$



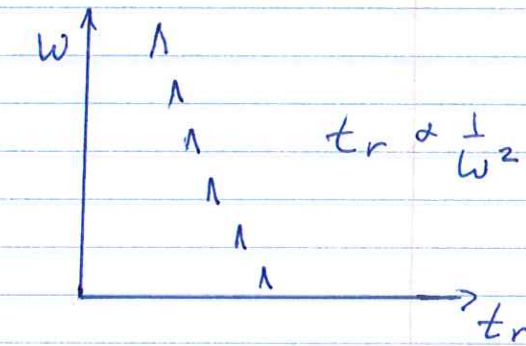
The envelope moves with the group velocity

$$v_{gr} = \frac{\Delta\omega}{\Delta k}$$

Pulsar Dispersion

Since the group velocity is a function of frequency, pulses arrive later for lower frequency ω.

Assume that $\omega \gg \omega_p$



$$v_{gr} = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$$

$$\approx c\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)$$

For a pulsar at distance d, the time for a pulse to reach the earth is:

$$t_r = \int_0^d \frac{d\ell}{v_{gr}} \approx \frac{1}{c}\int_0^d \left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) d\ell$$

$$= \frac{d}{c} + \frac{2\pi e^2}{mc\omega^2}\int_0^d n\, d\ell = \frac{d}{c} + \frac{2\pi e^2}{mc\omega^2} DM$$

where $DM \equiv \int_0^d n_e\, d\ell$ and n_e is the electron density.

The slope of the relation between t_r and ν is

$$\frac{dt_r}{d\nu} = 2\pi\frac{dt_r}{d\omega} = -\frac{8\pi^2 e^2}{mc\omega^3} DM = \frac{-e^2}{\pi mc\nu^3} DM$$

Example: For the crab pulsar, $d = 2000$ pc, $DM = 56.7$ pc cm^{-3} $(= 1.75\times10^{20}$ cm$^{-2})$

$DM = \langle n_e \rangle d$ so $\langle n_e \rangle = 0.028$ cm^{-3}

A typical observing frequency is 4×10^8 Hz. In this example

$$\frac{dt_r}{d\nu} = -7.3\times10^{-9}\ \frac{\text{s}}{\text{Hz}}, \quad dt_r = 0.033 \text{ s when } d\nu = 4.5\times10^6 \text{ Hz}$$

Faraday Rotation of the plane of polarization

If there is an ambient magnetic field that is stronger than the E and B fields in the electromagnetic wave, the electron will feel an additional force

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - e\frac{\vec{v}\times\vec{B}}{c}$$ where E is the wave field and B is the ambient field.

For simplicity assume that the wave is propagating in the direction of B, which we make the z-direction. A linearly polarized wave can be represented as the sum of two oppositely polarized circular waves

$$\vec{E}(t) = E_0 e^{-i\omega t}(\hat{x} \mp i\hat{y}) \qquad \begin{matrix}(-)\text{ right}\\(+)\text{ left}\end{matrix}$$

Now we will show by substitution that the solution to the force equation for the electron's velocity is:

$$\vec{v}(t) = v_0 e^{-i\omega t}(i\hat{x} \pm \hat{y}) \qquad \begin{matrix}(+)\text{ right}\\(-)\text{ left}\end{matrix}$$

which means that $\vec{v}(t)$ is $90°$ out of phase with $\vec{E}(t)$.

$$-i\omega v_0(i\hat{x} \pm \hat{y}) = -\frac{e}{m}E_0(\hat{x} \mp i\hat{y}) - \frac{ev_0B}{mc}(i\hat{x}\times\hat{z} \pm \hat{y}\times\hat{z})$$

$$\omega v_0(\hat{x} \mp i\hat{y}) = -\frac{e}{m}E_0(\hat{x} \mp i\hat{y}) \mp \frac{eBv_0}{mc}(\hat{x} \mp i\hat{y})$$

$$v_0\left(\omega \pm \frac{eB}{mc}\right) = -\frac{e}{m}E_0 \qquad \text{Note: } \frac{eB}{mc} = \omega_B$$

Thus $$\vec{v}(t) = -\frac{ie\vec{E}(t)}{m(\omega \pm \omega_B)} \qquad \begin{matrix}(+)\text{ right}\\(-)\text{ left}\end{matrix}$$

Since $\vec{J} = -en_e\vec{v} = \sigma\vec{E}$

$$\boxed{\sigma = \frac{ie^2n_e}{m(\omega \pm \omega_B)}}$$

Now the wavenumber can be expressed as

$$k^2 = \frac{\omega^2}{c^2} + i\,\frac{4\pi\sigma\omega}{c^2} = \frac{\omega^2}{c^2} - \frac{4\pi n_e e^2}{mc^2}\left(\frac{\omega}{\omega \pm \omega_B}\right)$$

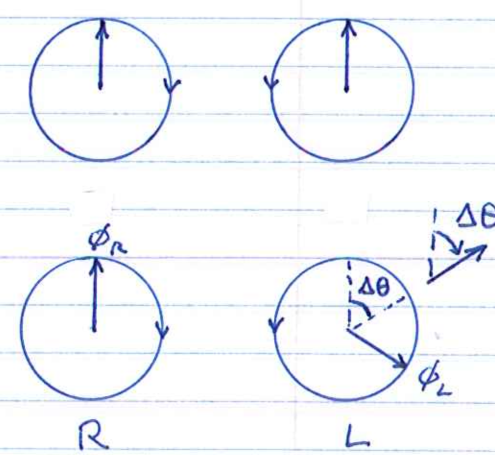
$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}}$$

In astronomy we are usually in the limits $\omega \gg \omega_p$ and $\omega \gg \omega_B$. Then

$$k_{R,L} \approx \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}\left(1 \mp \frac{\omega_B}{\omega}\right)\right] \qquad \begin{array}{ll} (-) & \text{right} \\ (+) & \text{left} \end{array}$$

The total phase angle ϕ through which the $\vec{E}$ vector rotates when the wave propagates a distance d is $\vec{k}\cdot\vec{d}$, or

$$\phi_{R,L} = \int_0^d k_{R,L}\, ds$$



Since ϕ is different for right and left circular polarization, the plane of polarization will rotate by an angle $\Delta\theta$,

$$\Delta\theta = \frac{1}{2}\int_0^d (k_R - k_L)\, ds$$

$$\Delta\theta = \frac{1}{2}\int_0^d \frac{\omega}{c}\,\frac{\omega_p^2}{\omega^2}\,\frac{\omega_B}{\omega}\, ds$$

$$\Delta\theta = \frac{1}{2c\omega^2}\int_0^d \frac{4\pi n_e e^2}{m}\,\frac{eB}{mc}\, ds$$

$$\Delta\theta = \frac{2\pi e^3}{(mc)^2\omega^2}\underbrace{\int_0^d n_e B_{\parallel}\, ds}_{\equiv \text{ RM (rotation measure)}}$$

The rotation angle depends on frequency. Measurements at more than one frequency are needed to determine the value of the integral, the RM.

$B_{\parallel}$ is the component of B in the direction of propagation.