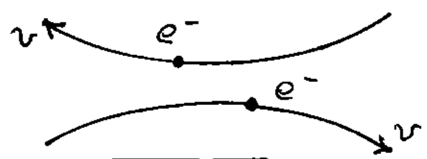
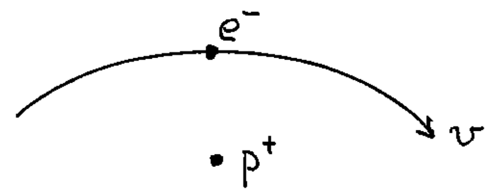
<u>Bremsstrahlung</u> (braking radiation, or free-free emission)

is the radiation due to the acceleration of an <u>unbound</u> charge in the Coulomb field of another charge. We will use the dipole approximation to calculate the radiation.

1. Collisions between like particles, $(e^- + e^-)$ or $(p^+ + p^+)$ do not radiate in the dipole approximation because the dipole moment $\sum_i q_i \vec{r}_i$ is a constant in time. But in collisions between electrons and protons, the electrons radiate the most because the accelerations are inversely proportional to mass



$\boxed{\ddot{d} = 0}$



$\boxed{\ddot{d} = -e\dot{v}}$

2. a) Classical formulae can be used in the case of distant encounters, in which the deviation from a straight line is small and the frequency of radiation is small.

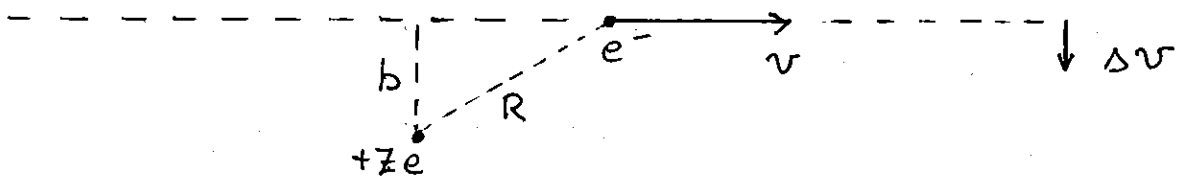
   b) Non-relativistic formulae can be used when $v \ll c$.

   c) When the collisions are close and the scattering angle is large, quantum calculations must be made (Gaunt factor).

3. Emission spectra from collections of particles can be calculated, such as "thermal bremsstrahlung" and "non-thermal bremsstrahlung"

4. Free-free <u>absorption</u> can occur at low frequencies. For thermal bremsstrahlung, the optically thick spectrum $\propto \nu^2$.

Start with a distant encounter in the rest frame of an ion, which is approximately the center-of-mass frame. The change in velocity of the electron, Δv, is much smaller than its v, and it is perpendicular to v in the small-angle scattering approximation. The "impact parameter" b is the distance of closest approach.



The dipole moment $\vec{d} = -e\vec{R}$, and $\ddot{\vec{d}} = -e\dot{\vec{v}}$. The radiation spectrum is proportional to the square of the Fourier transform of $d(t)$, which is $\hat{d}(\omega)$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\hat{d}(\omega)\right|^2 \qquad \left[\frac{erg}{Hz}\right]$$

To get an expression for $|\hat{d}(\omega)|^2$, first use the definition

$$d(t) = \int_{-\infty}^{\infty} \hat{d}(\omega)\, e^{-i\omega t}\, d\omega$$

Take the time derivative of this equation twice:

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 \hat{d}(\omega)\, e^{-i\omega t}\, d\omega$$

Therefore, the Fourier transform of $\ddot{d}$ is $-\omega^2\hat{d}(\omega)$. That is

$$-\omega^2\hat{d}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ddot{d}(t)\, e^{i\omega t}\, dt \qquad \text{where } \ddot{d}(t) = -e\dot{v}$$

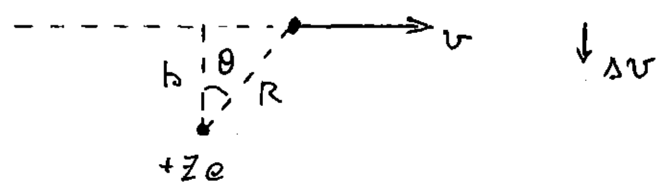
Now the problem is to evaluate this integral. The electron interacts strongly with the proton only during its closest approach, which lasts a time $\tau \approx b/v$, which is called the "collision time". In the frequency limit $\omega \ll 1/\tau$, the exponential in the integral is 1. In the opposite limit, $\omega \ll 1/\tau$, the exponential oscillates many times, so the integral is small.

$$\hat{d}(\omega) \simeq \begin{cases} 0 & \omega \gg 1/\tau \\ \dfrac{+e}{2\pi\omega^2}\displaystyle\int_{-\infty}^{\infty} dv = \dfrac{+e\,\Delta v}{2\pi\,\omega^2} & \omega \ll 1/\tau \end{cases}$$

$$\frac{dW}{d\omega} = \begin{cases} 0 & \omega \gg 1/\tau \\ \dfrac{2e^2}{3\pi c^3}\,|\Delta v|^2 & \omega \ll 1/\tau \end{cases}$$

How do we evaluate Δv? Since the deflection angle is small, Δv is mostly just the final component of v perpendicular to the path

$$\Delta v \approx \int_{-\infty}^{\infty} a_\perp\, dt \quad \text{and integrate the perpendicular acceleration}$$



$$= \frac{Ze^2}{m}\int_{-\infty}^{\infty} \frac{\cos\theta\, dt}{R^2}$$

$$= \frac{Ze^2}{m}\int_{-\infty}^{\infty} \frac{b}{R^3}\, dt$$

$$= \frac{Ze^2}{m}\int_{-\infty}^{\infty} \frac{b\, dt}{(b^2+v^2t^2)^{3/2}} = \frac{2Ze^2}{mbv}$$

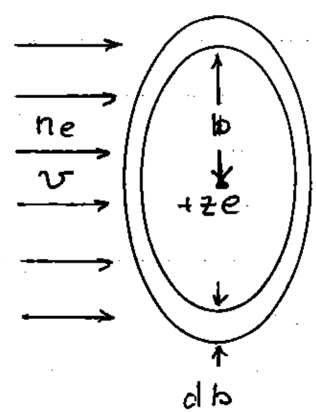
So the spectrum emitted by a single electron is

$$\frac{dW(b)}{d\omega} = \frac{8Z^2e^6}{3\pi c^3 m^2 b^2 v^2} \qquad \omega \ll \frac{v}{b}$$

Now consider a medium of electron density n_e and ion density n_i. For simplicity let the electrons have a single velocity v.

The flux of electrons is $n_e v$ [$cm^{-2}\,s^{-1}$]

The rate of electrons incident at impact parameter b is $2\pi b\,db\,n_e v$ [s^{-1}]



The collision rate per unit volume is $2\pi b\,db\,n_e n_i v$ [$cm^{-3}\,s^{-1}$]

The specific emissivity for the range of particular impact parameters $b \to b+db$ is $2\pi b\,db\,n_e n_i v \dfrac{dW(b)}{d\omega}$ $\left[\dfrac{erg}{cm^3\,Hz\,s}\right]$

The spectrum is then obtained by integrating over all impact parameters b.

$$\frac{dW}{d\omega\,dV\,dt} = 2\pi n_e n_i v \int_{b_{min}}^{b_{max}} \frac{dW(b)}{d\omega}\, b\,db \qquad \left[\frac{erg}{cm^3\,Hz\,s}\right]$$

Now we have to think about what is the appropriate value of b_{min}, and whether $b_{max} = \infty$ is valid. The limit on b_{max} can be estimated from the requirement that

$$\omega < v/b, \quad \text{or} \quad b_{max} = \frac{v}{\omega}$$

This is a good approximation since $dW(b)/d\omega$ decreases rapidly when $b > v/\omega$.

b_{min} can be estimated in two ways:

(1) b_{min} must be large enough that the small-deflection analysis is valid, namely, $\Delta v < v$. Since we showed that

$$\Delta v = \frac{2Ze^2}{mbv}$$

this occurs when $b_{min} = \frac{2Ze^2}{mv^2}$

(This is the same as the classical limit for an unbound orbit, one with $E = K + U = 0$.)

(2) b_{min} must satisfy the Heisenberg uncertainty principle,

$$\Delta x \, \Delta p \geq \hbar$$

Since the uncertainty in position is $\approx b_{min}$, and the uncertainty in momentum must be $\leq p$, we can say

$$b_{min} = \frac{\hbar}{mv}$$

When $b_{min}(1) > b_{min}(2)$ a classical analysis is valid

When $b_{min}(2) > b_{min}(1)$ a quantum calculation is needed

In general, the exact bremsstrahlung spectrum can be given in terms of a correction factor, $g_{ff}(v, \omega)$, the free-free Gaunt factor.

$$\frac{dW}{d\omega \, dV \, dt} = \frac{16\pi}{3\sqrt{3}} \frac{e^6}{c^3 m^2 v} n_e n_i Z^2 g_{ff}(v, \omega)$$

where $g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{max}}{b_{min}}\right)$

Thermal Bremsstrahlung

is obtained by averaging the spectrum for a single speed v over a thermal distribution of electrons. Instead of n_e, write

$$n_e(v)\, d^3\vec{v} = n_e(v)\, 4\pi v^2 dv$$

$$n_e(v)\, d^3\vec{v} = 4\pi n_e \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{m_e v^2}{kT}} v^2 dv$$

The distribution is normalized so that $\int_0^\infty n_e(v)\, d^3\vec{v} = n_e$.

The probability that an electron has a velocity in the range $v \to v+dv$ is:

$$4\pi v^2 dv \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{m v^2}{kT}}$$

Integrate this function times the spectrum that we had for a single electron velocity, and normalize

$$\frac{dW}{d\omega\, dV\, dt} = \frac{\dfrac{16\pi e^6 n_e n_i Z^2}{3\sqrt{3}\, c^3 m^2} \displaystyle\int_{v_{min}}^{\infty} \frac{4\pi v^2 dv}{v} e^{-\frac{1}{2}\frac{m v^2}{kT}} g_{ff}(v, \omega)}{\displaystyle\int_0^\infty 4\pi v^2 dv\, e^{-\frac{1}{2}\frac{m v^2}{kT}}}$$

Note the limit v_{min}, which is given by the requirement that $\frac{1}{2} m_e v^2 > \hbar\omega$ in order to create a photon of energy $\hbar\omega$. Therefore

$$v_{min} = \sqrt{2h\nu / m_e} \qquad \text{where } \nu = \omega/2\pi$$

Evaluating the integral requires integrating over the Gaunt factor. The result is

$$\frac{dW}{d\nu\, dV\, dt} = \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3 k m_e}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}(\nu, T)$$

where $g_{ff}(\nu, T)$ is the velocity-averaged Gaunt factor. The specific emissivity is

$$\boxed{\mathcal{E}_\nu = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}} \quad \left[\frac{erg}{cm^3\, s\, Hz}\right]$$

The total power integrated over frequency is

$$\boxed{\mathcal{E} = \int \mathcal{E}_\nu d\nu = 1.4 \times 10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_{ff}(T)} \quad \left[\frac{erg}{cm^3\, s}\right]$$

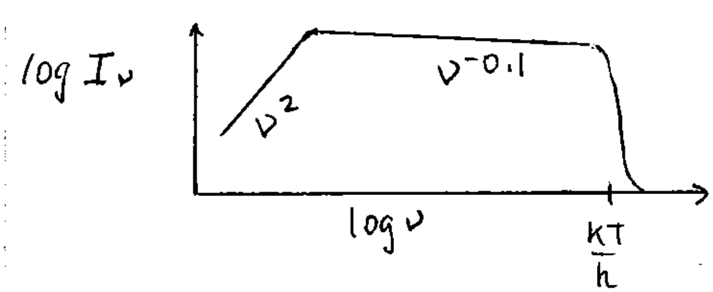
The frequency average Gaunt factor ranges from 1.1-1.5. $\bar{g}_{ff}(T) = 1.2$ is a good approximation for $T \gtrsim 10^6$ K

For an ionized plasma of "cosmic abundances" we must sum over ions

$$\sum_i n_e n_i Z^2 \approx 1.4\, n_H^2 \qquad \left\{ \begin{array}{ll} 90\%\ H & Z = 1 \\ 10\%\ He & Z = 2 \\ 1\%\ \text{else} & \end{array} \right.$$

Finally, $\mathcal{E} = 2.4 \times 10^{-27}\, T^{1/2} n_H^2$

Thermal bremsstrahlung is referred to as having an exponential spectrum, or a "flat" spectrum in a log-log plot, up to a cutoff $h\nu_{max} \approx kT$. However, because of the weak dependence of $g_{ff}(\nu, T)$ with frequency, the low-frequency (radio) spectrum has a slope of index ≈ -0.1



The optically thick (self absorbed) part of the spectrum is $\propto \nu^2$...

147

where $v_{\min} \equiv (2h\nu/m)^{1/2}$, and using $d\omega = 2\pi d\nu$, we obtain

$$\frac{dW}{dV\,dt\,d\nu} = \frac{2^5 \pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}. \qquad (5.14a)$$

Evaluating eq. (5.14) in CGS units, we have for the emission (erg s^{-1} cm^{-3} Hz^{-1})

$$\epsilon_\nu^{ff} \equiv \frac{dW}{dV\,dt\,d\nu} = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}. \qquad (5.14b)$$

Here $\bar{g}_{ff}(T,\nu)$ is a *velocity averaged Gaunt factor*. The factor $T^{-1/2}$ in Eq. (5.14) comes from the fact that $dW/dV\,dt\,d\omega \propto v^{-1}$ [cf. Eq. (5.11) and $\langle v \rangle \propto T^{1/2}$. The factor $e^{-h\nu/kT}$ comes from the lower-limit cutoff in the velocity integration due to photon discreteness and the Maxwellian shape for the velocity distribution.

Approximate analytic formulas for $\bar{g}_{ff}$ in the various regimes in which large-angle scatterings and small-angle scatterings are dominant, in which

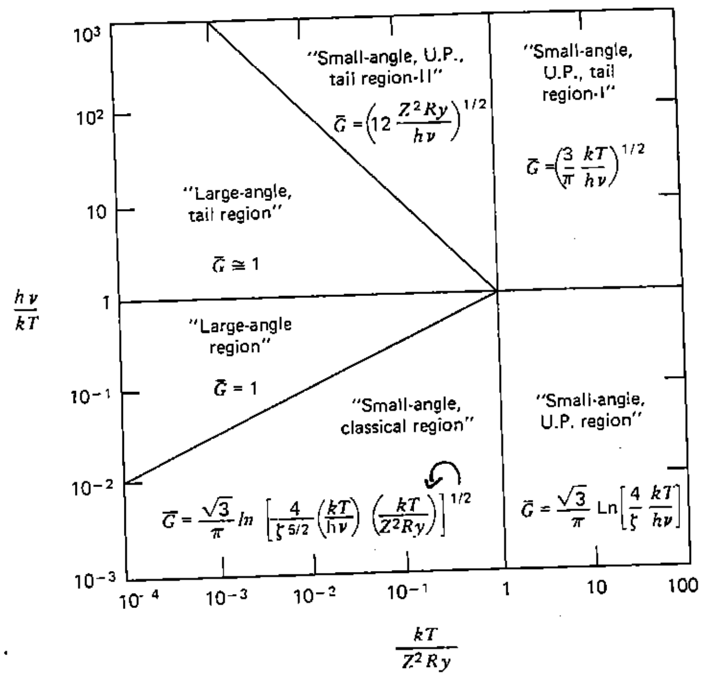


Figure 5.2 *Approximate analytic formulae for the gaunt factor $\bar{g}_{ff}(\nu,T)$ for thermal bremsstrahlung. Here $\bar{g}_{ff}$ is denoted by $\bar{G}$ and the energy unit Ry = 13.6 eV. (Taken from Novikov, I. D. and Thorne, K. S. 1973 in Black Holes, Les Houches, Eds. C. DeWitt and B. DeWitt, Gordon and Breach, New York.)*

$\zeta = \frac{1}{0.577}$

the uncertainty principle (U. P.) is important in the minimum impact parameter, and so on are indicated in Fig. 5.2. Figure 5.3 gives numerical graphs of $\bar{g}_{ff}$. The values of $\bar{g}_{ff}$ for $u \equiv h\nu/kT \gg 1$ are not important, since the spectrum cuts off for these values. Thus $\bar{g}_{ff}$ is of order unity for $u \sim 1$ and is in the range 1 to 5 for $10^{-4} < u < 1$. We see that good order of magnitude estimates can be made by setting $\bar{g}_{ff}$ to unity.

We also see that bremsstrahlung has a rather "flat spectrum" in a log-log plot up to its cutoff at about $h\nu \sim kT$. (This is true only for optically thin sources. We have not yet considered *absorption* of photons by free electrons.)

To obtain the formulas for *nonthermal bremsstrahlung*, one needs to know the actual distributions of velocities, and the formula for emission from a single-speed electron must be averaged over that distribution. To do this one also must have the appropriate Gaunt factors.

Let us now give formulas for the total power per unit volume emitted by thermal bremsstrahlung. This is obtained from the spectral results by integrating Eq. (5.14) over frequency. The result may be stated as

$$\frac{dW}{dt\,dV} = \left(\frac{2\pi kT}{3m}\right)^{1/2} \frac{2^5 \pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}_B, \qquad (5.15a)$$

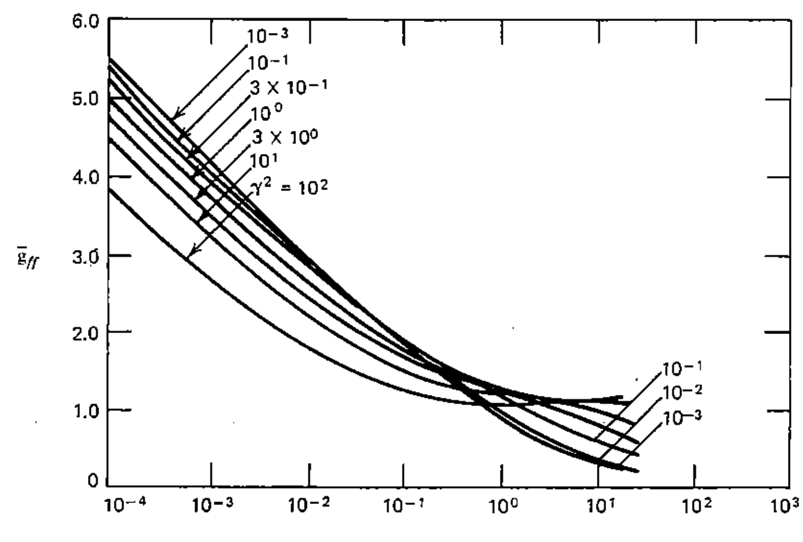


Figure 5.3 *Numerical values of the gaunt factor $\bar{g}_{ff}(\nu,T)$. Here the frequency variable is $u = 4.8 \times 10^{11}\nu/T$ and the temperature variable is $\gamma^2 = 1.58 \times 10^5 Z^2/T$. (Taken from Karzas, W. and Latter, R. 1961, Astrophys. J. Suppl., 6, 167.)*

$u = 4.8 \times 10^{-11}\,\nu/T$

Free-free absorption

Thermal bremsstrahlung is thermal radiation so its absorption coefficient is easily calculated once its emission coefficient is known. We just derived the latter,

$$\varepsilon_\nu = \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3k m_e}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}(\nu, T)$$

where $\varepsilon_\nu = 4\pi j_\nu$

and $\dfrac{j_\nu}{\alpha_\nu} = B_\nu(T) = \dfrac{2h\nu^3}{c^2} \dfrac{1}{e^{h\nu/kT} - 1}$

Therefore,

$$\alpha_\nu = \frac{\varepsilon_\nu}{4\pi B_\nu(T)} = \frac{4e^6}{3m_e hc} \left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} \frac{n_e n_i}{\nu^3} \left(1 - e^{-h\nu/kT}\right) \bar{g}_{ff}$$

Absorption is usually negligible unless $h\nu \ll kT$, in which case

$$\alpha_\nu = 0.018\, T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff}(\nu, T)$$

Remember that the specific intensity approaches the source function in the limit of large optical depth. For bremsstrahlung, this occurs at low frequency, where $I_\nu \to j_\nu/\alpha_\nu \propto \nu^2$, the Rayleigh-Jeans limit for a source of uniform temperature.