Spectrum of Synchrotron Radiation $P(\omega)$

$$P(\omega) = \frac{\sqrt{3}}{2\pi} \frac{e^3 B \sin\alpha}{mc^2} F(x)$$

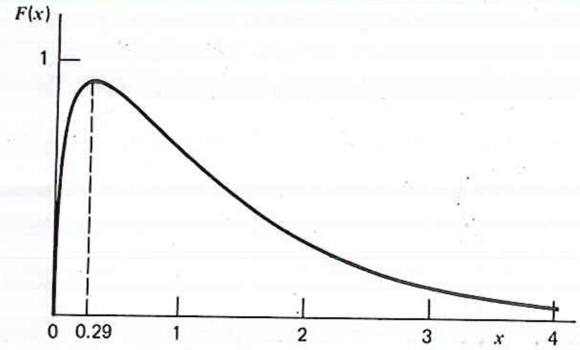


Figure 6.6 Function describing the total power spectrum of synchrotron emission. Here $x = \omega/\omega_c$. (Taken from Ginzburg, V. and Syrovatskii, S. 1965, Ann. Rev. Astron. Astrophys., 3, 297.)

where $x = \omega/\omega_c$

$$\omega_c \equiv \frac{3}{2} \gamma^3 \omega_B \sin\alpha$$

$$\omega_B = \frac{eB}{\gamma mc}$$

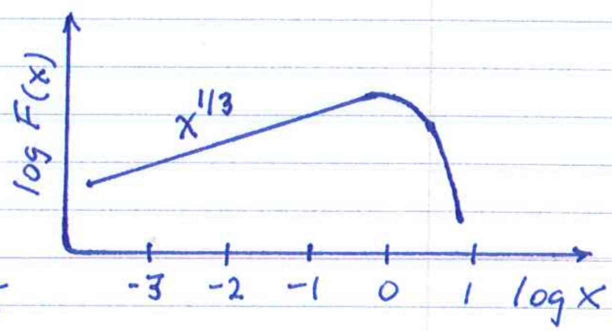
Look at some properties of $F(x)$ before deriving it:

$$\int_0^\infty F(x)\,dx = \Gamma\left(\tfrac{7}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{8\pi}{9\sqrt{3}} \quad \text{from Gamma functions}$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$



Limits for large and small x

$$F(x) \approx \sqrt{\frac{\pi}{2}}\, x^{1/2} e^{-x} \qquad x \gg 1$$

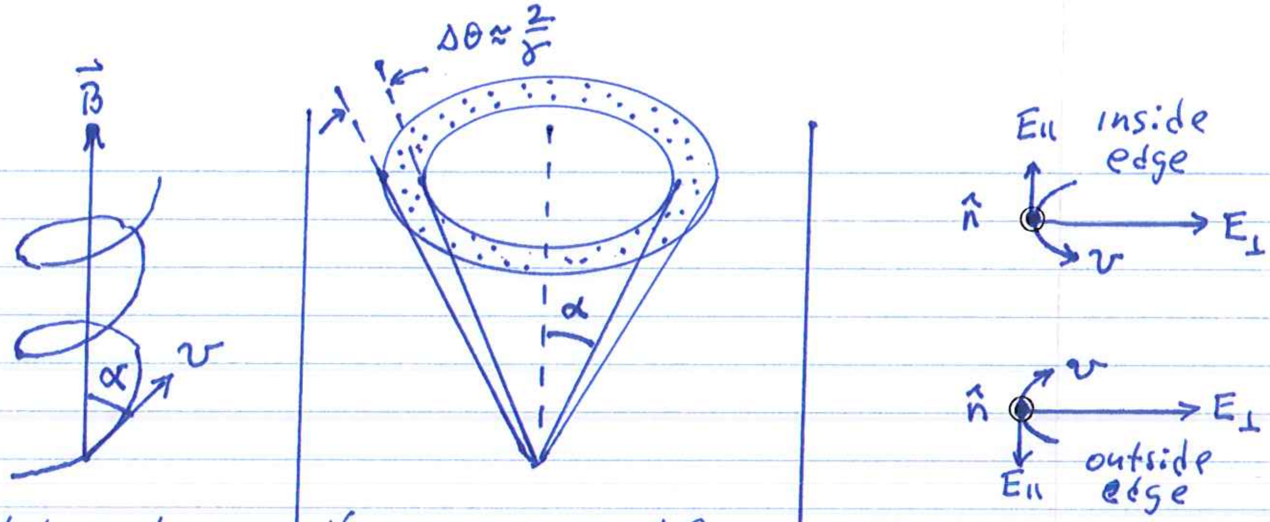
$$F(x) \approx \frac{4\pi}{\sqrt{3}\,\Gamma(\frac{1}{3})} \left(\frac{x}{2}\right)^{1/3} \qquad x \ll 1$$

Total Power $P = \int_0^\infty P(\omega)\,d\omega = \boxed{\frac{2}{3}\frac{e^2}{c^3}\gamma^4\left(\frac{e v_\perp B}{\gamma mc}\right)^2}$ from page 102

But $P = \frac{\sqrt{3}}{2\pi} \frac{e^3 B \sin\alpha}{mc^2} \omega_c \int_0^\infty F\left(\frac{\omega}{\omega_c}\right)\frac{d\omega}{\omega_c}$ since $dx = d\omega/\omega_c$

$$P = \frac{\sqrt{3}}{2\pi} \frac{e^3 B \sin\alpha}{mc^2} \frac{3}{2}\gamma^3 \frac{eB}{\gamma mc}\sin\alpha \frac{8\pi}{9\sqrt{3}} = \boxed{\frac{2}{3}\frac{e^2\gamma^4}{c^3}\left(\frac{e B_\perp}{\gamma m}\right)^2}$$

This explains the normalizing factor in front of $F(x)$ if we let $v/c \approx 1$ so $B\sin\alpha \approx B v_\perp/c \equiv B_\perp$.



Pitch angle α | Visibility cone $\Delta\theta$ | Observer's view

The spectrum of synchrotron radiation can be derived by considering the contributions of the parallel and perpendicular polaritations separately, in terms of modified Bessel functions.

$$\frac{dW_{\perp}}{d\omega} = \frac{\sqrt{3}}{2c} e^2 \gamma \sin\alpha \left[F(x) + G(x) \right] \qquad \left[\frac{erg}{Hz}\right]$$

$$\frac{dW_{\parallel}}{d\omega} = \frac{\sqrt{3}}{2c} e^2 \gamma \sin\alpha \left[F(x) - G(x) \right]$$

where $F(x) = x \int_x^{\infty} K_{5/3}(z)\,dz$ and $G(x) = x K_{2/3}(x)$

$$K_p(x) = \frac{\pi}{2} \frac{I_{-p}(x) - I_p(x)}{\sin(p\pi)} \quad \text{where } I_p(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+p}}{k!\,(k+p)!}$$

<u>Power per unit frequency</u> $\left[\frac{erg}{s\ Hz}\right]$

$$P(\omega) = \frac{1}{T}\left[\frac{dW_{\perp}}{d\omega} + \frac{dW_{\parallel}}{d\omega}\right] = \frac{\omega_B}{2\pi} \frac{\sqrt{3}}{c} e^2 \gamma \sin\alpha\, F(x)$$

$$P(\omega) = \frac{\sqrt{3}\, e^3 B \sin\alpha}{2\pi m c^2} F(x) \qquad \omega_B = \frac{eB}{\gamma m c}$$

Since $P(\nu)\,d\nu = P(\omega)\,d\omega$, we can also use $d\omega = 2\pi\, d\nu$

$$P(\nu) = \frac{\sqrt{3}\, e^3 B \sin\alpha}{m c^2} F\left(\frac{\nu}{\nu_c}\right)$$

Polarization of Synchrotron Radiation

At the instant the electron is moving toward the observer the observed $\vec{E}$ field is perpendicular to the projection of $\vec{B}$ on the sky. When the electron's velocity is not precisely pointing at the observer, there will also be a parallel component of $\vec{E}$. In general the polarization is elliptical with helicity depending on whether the observer is just inside or outside the cone of maximum intensity. If there is a distribution of pitch angles, the observer will be inside and outside different cones, the circular polarization components will cancel, and the radiation will be <u>linearly polarized perpendicular to $\vec{B}$</u>.

$$\Pi(\omega) = \frac{P_\perp(\omega) - P_\parallel(\omega)}{P_\perp(\omega) + P_\parallel(\omega)} = \frac{F(x) + G(x) - (F(x) - G(x))}{2F(x)}$$

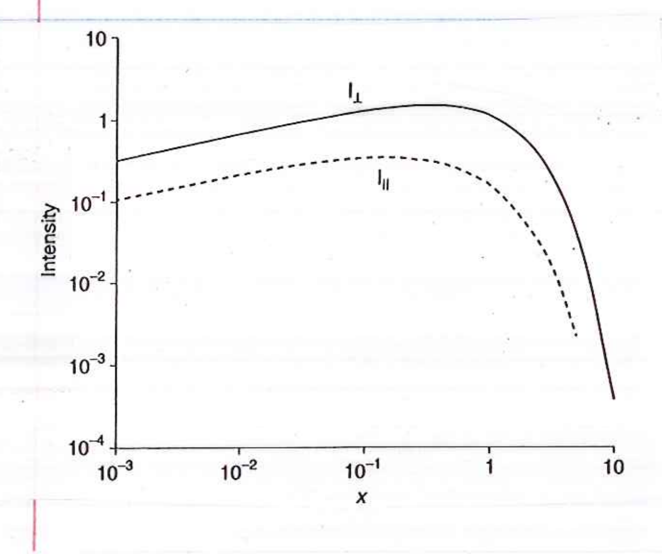
$$\Pi(\omega) = \frac{G(x)}{F(x)}$$

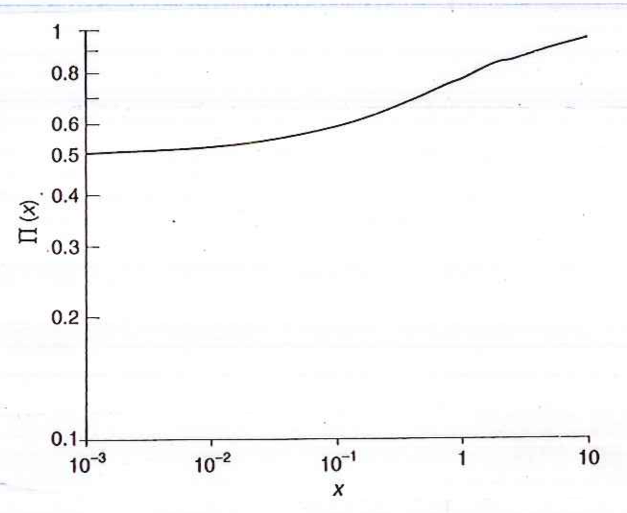
The degree of polarization depends on frequency through the different functions $F(x)$ and $G(x)$.

The degree of polarization averaged over all frequencies is:

$$\Pi = \frac{\int G(x)\,dx}{\int F(x)\,dx} = \frac{\Gamma(4/3)\,\Gamma(2/3)}{\Gamma(7/3)\,\Gamma(2/3)} = \frac{\Gamma(4/3)}{\Gamma(7/3)} = \frac{3}{4} = 75\%$$

The functions $P_\perp(x)$, $P_\parallel(x)$, and $\Pi(x)$ are shown here:





<u>Synchrotron Radiation from a power-law distribution of electron energies.</u> What is it's spectrum?

Let the electron energies be denoted by γ, their number density per unit γ by $n(\gamma)$, and the power-law index by p.

$$n(\gamma)\,d\gamma = n_0 \gamma^{-p}\,d\gamma \qquad \text{for } \gamma_1 < \gamma < \gamma_2$$

The total electron density is $n = \int_{\gamma_1}^{\gamma_2} n(\gamma)\,d\gamma$ $[cm^{-3}]$

The specific emissivity is $\mathcal{E}_\nu$, where $\mathcal{E}_\nu = \int P(\nu)\,n(\gamma)\,d\gamma$

$$\mathcal{E}_\nu = n_0 \int_{\gamma_1}^{\gamma_2} 2\pi\, P(\omega)\,\gamma^{-p}\,d\gamma \qquad \left[\frac{erg}{cm^3\, s\, Hz}\right]$$

Let's try to write this spectrum as a function of frequency ω (or ν). We need

$$P(\omega) = \frac{\sqrt{3}\,e^3}{2\pi mc^2} B \sin\alpha\, F(x) \qquad x = \frac{\omega}{\omega_c}$$

$$\omega_c = \frac{3}{2}\gamma^2 \frac{eB}{mc}\sin\alpha$$

Let's drop physical constants and find the dependence of $\mathcal{E}_\nu$ on physical variables n_0, B, p, and ν

$$\mathcal{E}_\nu \propto n_0 B \int_{\gamma_1}^{\gamma_2} F(\omega/\omega_c)\,\gamma^{-p}\,d\gamma$$

Write the integrand in terms of ω instead of γ.
To do this, use $\omega_c \propto \gamma^2 B$, $d\omega_c \propto 2\gamma\, d\gamma\, B$

Then $\gamma^{-p} \propto \left(\frac{B}{\omega_c}\right)^{p/2}$ and $d\gamma \propto \frac{1}{2}\frac{d\omega_c}{\gamma B} \propto \frac{1}{2}\left(\frac{B}{\omega_c}\right)^{1/2}\frac{d\omega_c}{B}$

$$\Rightarrow \gamma^{-p}\,d\gamma \propto \omega_c^{-\frac{1}{2}(p+1)}\, B^{\frac{1}{2}(p-1)}\, d\omega_c$$

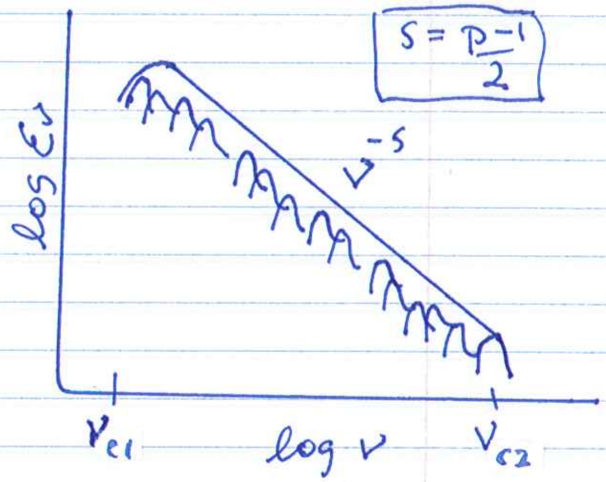
and $$\mathcal{E}_\nu \propto n_0 B^{\frac{1}{2}(p+1)} \int_{\omega_{c1}}^{\omega_{c2}} F(\omega/\omega_c)\,\omega_c^{-\frac{1}{2}(p+1)}\,d\omega_c$$

where ω_{c1} and ω_{c2} are the critical frequencies corresponding to γ_1 and γ_2.

Since $\frac{\omega}{\omega_c} = x$, $d\omega_c = -\frac{\omega}{x^2}dx$

$$\mathcal{E}_\nu \propto n_0 B^{\frac{1}{2}(p+1)} \omega^{-\frac{1}{2}(p-1)} \int_{x_1}^{x_2} F(x)\, x^{\frac{1}{2}(p-3)}\, dx$$

Now we will approximate the integral with a definite integral with limits 0 and ∞ because the function F is highly peaked around the critical frequency.



Let $x_1 \to 0$, $x_2 \to \infty$

Having done that,

the integral is a constant that depends only on p.

Putting back all the physical constants gives

$$\boxed{\mathcal{E}_\nu = 1.7\times10^{-21}\, n_0\, a(p)\, B^{s+1}\left(\frac{\nu}{4\times10^6}\right)^{-s} \frac{\text{erg}}{\text{cm}^3\ \text{s Hz}}}$$

where $a(p)$ is a function of order unity:

p	1	1.5	2	2.5	3	5
$a(p)$	0.283	0.147	0.103	0.0852	0.0742	0.0922

Also, $\sin\alpha$ has been averaged over an isotropic distr.

* Moral: The spectrum of synchrotron radiation from a power-law distribution of electron energies with index p, is a power-law in frequency with index $s = (p-1)/2$.
Also, polarization now depends only on p

$$\Pi = \frac{\int G(x)\,\gamma^{-p}\,d\gamma}{\int F(x)\,\gamma^{-p}\,d\gamma} = \frac{p+1}{p+7/3}$$

(See R+L Eq. 6.35)

$$\omega_c = \frac{3}{2} \gamma^3 \omega_B \sin\alpha$$

Summary of results $\quad U_{mag} = \frac{B^2}{8\pi} \quad \omega_B = \frac{eB}{\gamma mc}$

$$P_{synch} = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 \left(\frac{B^2}{8\pi}\right) \quad \left[\frac{erg}{s}\right]$$

averaged over pitch angles α

$$\nu_c = \frac{\omega_c}{2\pi} = \frac{3}{4\pi} \gamma^2 \frac{eB}{mc} \sin\alpha = 4.2\times10^6 \gamma^2 B \sin\alpha \quad [Hz]$$

Synchrotron lifetime $\quad t_s = \frac{\gamma mc^2}{P_{synch}} = \frac{3\times10^7}{\gamma U_{mag}} \quad [s]$

Alternatively, as a function of ν $\quad t_s = \frac{5\times10^{11}}{\nu_c^{1/2} (B\sin\alpha)^{3/2}} \quad [s]$

The synchrotron lifetime can be written either as a function of the electron's γ, or the frequency ν at which it emits, which is approximated as ν_c.

$$\mathcal{E}_\nu = 1.7\times10^{-21} n_0 a(p) B^{s+1} \left(\frac{\nu}{4\times10^6}\right)^{-s} \quad \left[\frac{erg}{cm^3\, s\, Hz}\right]$$

$s = \frac{p-1}{2} \quad \Pi = \frac{p+1}{p+7/3} \quad$ Results for power-law spectrum of electrons

One would like to use the synchrotron spectrum to diagnose physical conditions such as the strength of the magnetic field, the number density, and total energy in particles. But the observable quantities depend on both n and B, or on γ and B. So it is difficult to measure these quantities separately.

In the following sections, we discuss how observations of <u>synchrotron self-absorption</u>, or using an assumption of <u>equipartition of energy</u>, can separately diagnose the energy in electrons and magnetic field.

Synchrotron Self-Absorption

Remember that for *thermal* emission, the source function is the Planck spectrum, and the absorption coefficient can be calculated from the emission coefficient as

$$\alpha_\nu = \frac{j_\nu}{B_\nu(T)} = \frac{\mathcal{E}_\nu}{4\pi B_\nu(T)} = \frac{\mathcal{E}_\nu c^2}{8\pi\nu^2 kT} \quad \text{in the Rayleigh-Jean limit}$$

For a *power-law* distribution of electron energies, R. + L. §6.8 calculates the absorption coefficient in detail, finding

$$\alpha_\nu \propto (B \sin\alpha)^{\frac{p}{2}+1} \, \nu^{-(\frac{p}{2}+2)}$$

The source function in the case of a power law is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{\mathcal{E}_\nu}{4\pi\alpha_\nu} \propto \frac{B^{s+1}\,\nu^{-s}}{B^{\frac{p}{2}+1}\,\nu^{-(p/2+2)}} \quad \text{where } s = \frac{p-1}{2}$$

$$S_\nu \propto B^{-1/2}\,\nu^{+5/2} \quad (\underline{\text{not}}\ \nu^2 \text{ as for thermal emission})$$

However, the spectrum of a self-absorbed synchrotron source can be understood without calculating the absorption coefficient in detail. Just note that, at low frequencies, where absorption is most important, an optically thick source cannot be brighter than the Rayleigh-Jeans spectrum,

$$B_\nu = 2kT_B\,\nu^2/c^2$$

For a nonthermal source, the trick is to define a brightness temperature at each frequency ν using the *nonthermal* energy γmc^2 of the electron emitting at that frequency.

Let $kT_B = \gamma mc^2$ and $\nu = \frac{3}{4\pi}\gamma^2 \frac{eB}{mc}$

Eliminating γ, T_B is now a function of frequency ν

$$kT_B = \left(\frac{4\pi}{3}\frac{\nu mc}{eB}\right)^{1/2} mc^2 \approx 2mc^2\left(\frac{\nu mc}{eB}\right)^{1/2}$$

and $B_\nu = 2kT_B \frac{\nu^2}{c^2} = 4m\nu^2\left(\frac{\nu mc}{eB}\right)^{1/2} \propto \nu^{5/2}$

This shows the dependence of the brightness limit on $\nu^{5/2}$. Synchrotron self-absorption allows us to infer information about the size of a source even if it is not directly resoluable. Consider a spherical, self-absorbed source of radius R at distance d. The flux at Earth has an upper limit because it cannot exceed the following:

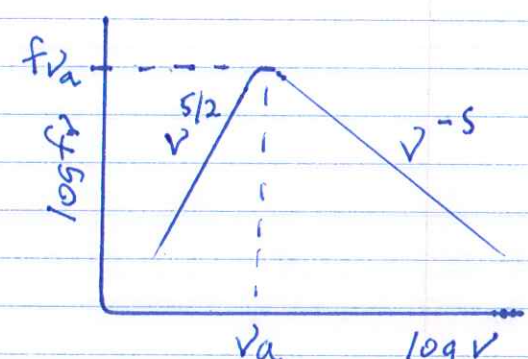
$$f_\nu \leq \pi B_\nu \frac{R^2}{d^2} \quad \left[\frac{erg}{cm^2\, s\, Hz}\right]$$

Replacing B_ν with its non-thermal analog

$$f_\nu \leq \frac{4\pi m^{3/2} c^{1/2} \nu^{5/2}}{(eB)^{1/2}}\left(\frac{R}{d}\right)^2$$

For an intrinsic power-law spectrum of the form ν^{-s}, there will be a self-absorption frequency ν_a at which the flux, f_{ν_a}, is a maximum. Below this frequency, $f_\nu \propto \nu^{5/2}$

If the quantities ν_a and f_{ν_a} are observed, the radius of the source can be expressed as



$$R = 6\times10^{14} f_{\nu_a}^{1/2} B^{1/4} \nu_a^{-5/4} d \quad [cm]$$

If the source is resolved so that its angular diameter $\theta = 2R/d$ is measured, then B can be inferred:

$$\theta = 2.5\times10^{-3}\left(\frac{f_{\nu_a}}{10^{-26}}\right)^{1/2} B^{1/4}\left(\frac{\nu_a}{10^8}\right)^{-5/4} \quad [arcsec]$$

Equipartition of Energy $U_{rel} = U_{mag}$

Even if a synchrotron source is not ~~resolved or~~ absorbed, its B-field strength can be estimated under the assumption of equipartition between the energy density in relativistic particles and in magnetic fields. Equipartition is an assumption, not a law. However, the total energy in particles and B-field is a minimum when equipartition holds. Here is the proof.

Assume a power-law distribution of electron energies

$$n(\gamma)\,d\gamma = n_0\,\gamma^{-p}\,d\gamma \qquad \gamma_{min} < \gamma < \gamma_{max}$$

$$U_{rel} = n_0 \int_{\gamma_{min}}^{\gamma_{max}} \gamma mc^2\, \gamma^{-p}\, d\gamma \qquad [\mathrm{erg/cm^3}]$$

Replace γ with characteristic frequency $\nu = \frac{3}{4\pi}\gamma^2 \frac{eB}{mc}$

Then $\gamma^{-p} = \left(\frac{4\pi mc}{3eB}\right)^{-p/2} \nu^{-p/2}$ and $\gamma\, d\gamma = \frac{2\pi mc}{3eB}\, d\nu$

$$U_{rel} = n_0 mc^2\, \frac{1}{2} \left(\frac{4\pi mc}{3eB}\right)^{1-p/2} \int_{\nu_{min}}^{\nu_{max}} \nu^{-p/2}\, d\nu$$

$$U_{rel} = \mathrm{const.}\ \frac{n_0\, B^{p/2-1}}{p/2-1} \left[\frac{1}{\nu_{min}^{\,p/2-1}} - \frac{1}{\nu_{max}^{\,p/2-1}}\right]$$

If $\nu_{max} \gg \nu_{min}$ and $p > 2$, then

$$U_{rel} \approx \frac{\mathrm{const.}\ n_0}{p/2-1} \left(\frac{B}{\nu_{min}}\right)^{p/2-1}$$

Since $p = 2s+1$, where s is the radiation spectral index

$$U_{rel} \approx \frac{\mathrm{const.}\ n_0}{s-1/2} \left(\frac{B}{\nu_{min}}\right)^{s-\frac{1}{2}} \qquad \text{as long as } s > 0.5$$

Now evaluate the synchrotron emissivity at ν_{min}.

$$\mathcal{E}_\nu = 1.7\times10^{-21}\, n_0\, a(p)\, B^{s+1}\left(\frac{4\times10^6}{\nu}\right)^s \quad \left[\frac{\text{erg}}{\text{cm}^3\,\text{s}\,\text{Hz}}\right]$$

$$\mathcal{E}(\nu_{min}) = \text{const}\cdot n_0 B^{s+1}\nu_{min}^{-s}$$

Now combine expressions for U_{rel} and $\mathcal{E}(\nu_{min})$ to eliminate n_0, the normalization of the particle distribution

$$U_{rel} = \frac{\text{const.}}{s-1/2}\,\frac{\mathcal{E}(\nu_{min})}{B^{3/2}}\,\nu_{min}^{-1/2}$$

The total energy is $U_{tot} = U_{rel} + U_{mag} = \frac{\text{const.}}{B^{3/2}} + \frac{B^2}{8\pi}$

Minimize U_{tot} with respect to B

$$\frac{dU_{tot}}{dB} = -\frac{3}{2}\frac{U_{rel}}{B} + \frac{B}{4\pi} = 0$$

$$U_{rel} = B^2/6\pi = \tfrac{4}{3}\,U_{mag} \qquad \text{Q.E.D.}$$

In practice, the particle energies and B-field energies are equated under the assumption of equipartition, as follows:

$$n_0\int_{\gamma_{min}}^{\gamma_{max}} \gamma mc^2\,\gamma^{-p}\,d\gamma = \frac{4}{3}\frac{B^2}{8\pi}$$

Solve this for n_0

$$n_0 \approx \frac{(p-2)\,\gamma_{min}^{p-2}}{mc^2}\,\frac{B^2}{6\pi}$$

Substitute ν_{min} for γ_{min} using $\nu = \frac{3}{4\pi}\gamma^2\frac{eB}{mc}$

Finally, insert the expression for n_0 into the one for $\mathcal{E}(\nu_{min})$ and solve for B. The result is:

$$B \propto \nu_{min}^{1/7}\, F_{\nu_{min}}^{2/7}\,(2s-1)^{2/7}\,(d^2/R^3)^{2/7}$$