Synchrotron Radiation

The Lorentz force law in covariant form is

4-force $$F^{\mu} = m_0 a^{\mu} = \frac{dp^{\mu}}{d\tau} = \frac{q}{c} F^{\mu\nu} u_{\nu}$$

field tensor: (row μ, column ν)
$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

4-velocity
$$u_{\nu} = (-\gamma_u c, \gamma_u \vec{u})$$

4-momentum
$$p^{\mu} = (\gamma_u m_0 c, \gamma_u m_0 \vec{u})$$

proper time $d\tau = \frac{dt}{\gamma_u}$

$\mu = 0$ $$\gamma_u \frac{d}{dt}(\gamma_u m_0 c) = \frac{q}{c}\gamma_u \vec{E}\cdot\vec{u}$$

$$\boxed{\frac{d}{dt}(\gamma_u m_0 c^2) = q\,\vec{E}\cdot\vec{u}}$$

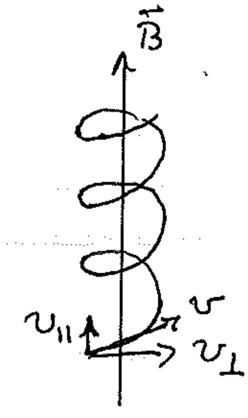
$\mu = 1,2,3$ $$\gamma_u \frac{d}{dt}(\gamma_u m_0 \vec{u}) = \frac{q}{c}(\gamma_u c\vec{E} + \gamma_u \vec{u}\times\vec{B})$$

$$\boxed{\frac{d}{dt}(\gamma_u m_0 \vec{u}) = q\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right)}$$

Conclusion: The Lorentz force law for a relativistic particle can be obtained by replacing m with γm_0.

Let $v = u$, the velocity of the particle in the lab. In a uniform magnetic field with no $\vec{E}$ field, $|v|$ is constant (until the charge radiates).

$$\gamma m_0 \frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B} \qquad v_{\parallel} = \text{const.}$$



$$a_{\perp} = \left|\frac{d\vec{v}_{\perp}}{dt}\right| = \frac{q\,|\vec{v}\times\vec{B}|}{\gamma m_0 c} = \frac{q v_{\perp} B}{\gamma m_0 c}$$

Acceleration is perpendicular to velocity

$a_\perp = \frac{v_\perp^2}{r} = \omega_B v_\perp$

ω_B = "angular velocity" or "angular frequency" or "gyrofrequency"

gyrofrequency: $\omega_B v_\perp = \frac{q v_\perp B}{\gamma m_0 c}$ $\quad \boxed{\omega_B = \frac{qB}{\gamma m_0 c}}$

gyroradius: $r_g = \frac{v_\perp}{\omega_B} = \boxed{\frac{\gamma m_0 v_\perp c}{qB}}$

Total radiated power is

$$P = \frac{2}{3}\frac{q^2}{c^3}\left(a_\perp'^2 + a_\parallel'^2\right) = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

$$= \frac{2}{3}\frac{q^2}{c^3}\gamma^4\left(\frac{q v_\perp B}{\gamma m_0 c}\right)^2$$

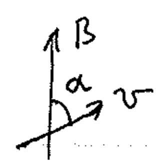
Let $q = e$

Classical electron radius $r_0 = \frac{e^2}{m_e c^2}$. For an electron

$$\boxed{P_{synch} = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2}$$

$\beta_\perp = \frac{v_\perp}{c}$

If there is a distribution of velocities that is isotropic over directions, average this formula over $\beta_\perp^2$. Let $\beta_\perp B = \beta B \sin\alpha$, where α is the pitch angle



$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi}\int \sin^2\alpha \, d\Omega$$

$$= \frac{\beta^2}{2}\int_{-1}^{1}(1-\mu^2)\,d\mu = \frac{2\beta^2}{3}$$

isotropic: $P_{synch} = \left(\frac{2}{3}\right)^2 r_0^2 c \beta^2 \gamma^2 B^2$

$$\boxed{P_{synch} = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 \left(\frac{B^2}{8\pi}\right)}$$

where $\sigma_T = \frac{8\pi r_0^2}{3}$

Now what happens as the charge loses energy?

$$\frac{d}{dt}(\gamma m_e c^2) = -\frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

For a relativistic particle assume that $\beta = 1$ and write $\beta_\perp B = B_\perp$.

$$\frac{d\gamma}{dt} = -\frac{2}{3}\frac{r_0^2}{m_e c} B_\perp^2 \gamma^2$$

$$\int_{\gamma_0}^{\gamma} \frac{d\gamma}{\gamma^2} = -A\int_0^t dt \qquad \text{where } A = \frac{2}{3}\frac{e^4}{m_e^3 c^5} B_\perp^2$$

$$\frac{1}{\gamma_0} - \frac{1}{\gamma} = -At \qquad \gamma_0 \text{ is the initial } \gamma$$

$$\boxed{\gamma = \frac{\gamma_0}{1 + A\gamma_0 t}}$$

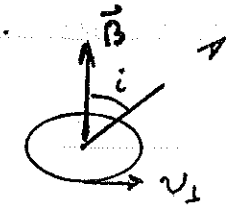
The time it takes to lose ½ the energy is $t_{1/2} = (A\gamma_0)^{-1}$

$$t_{1/2} = \frac{3 m_e^3 c^5}{2 e^4}\frac{1}{\gamma_0 B_\perp^2} \qquad \boxed{t_{1/2} = \frac{5.1\times 10^8}{\gamma_0 B_\perp^2}\ \text{s}}$$

Typical examples	γ	B	$t_{1/2}$ (s)	r_g (cm)	
interstellar medium	1000	3×10^{-6}	6×10^{16}	5.7×10^{11}	
radio galaxy lobe	1000	10^{-5}	5×10^{15}	1.7×10^{11}	
pulsar magnetosphere	10^7	10^9	5×10^{-17}	17	?!

Next, we want to examine the angular distribution of synchrotron radiation, and its spectrum. But first let's look at the non-relativistic limit, which is cyclotron radiation.

$$\frac{dP}{d\Omega} = \frac{(e v_\perp \omega)^2}{8\pi c^3}(1 + \cos^2 i)$$

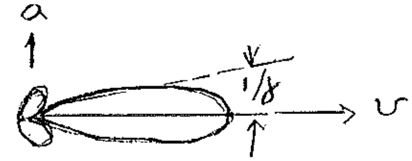


In the limit $\beta \to 1$, $\beta = (1 - \frac{1}{\gamma^2})^{1/2} \approx 1 - \frac{1}{2\gamma^2}$
and $\cos\theta \approx 1 - \theta^2/2$, so $\beta\mu \approx 1 - \theta^2/2 - \frac{1}{2\gamma^2}$

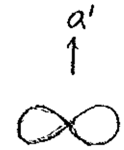
$$\frac{d\mathbb{P}_r}{d\Omega} = \frac{1}{\gamma^4}\frac{1}{(1-\beta\mu)^4}\frac{d\mathbb{P}_r'}{d\Omega'} \approx \frac{1}{\gamma^4}\frac{1}{(1-1+\theta^2/2+\frac{1}{2\gamma^2})^4}\frac{d\mathbb{P}'}{d\Omega'}$$

$$\frac{d\mathbb{P}_r}{d\Omega} \approx \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4 \frac{d\mathbb{P}'}{d\Omega'}$$

Maximum at $\theta = 0$ and decreases rapidly for $\theta > 1/\gamma$

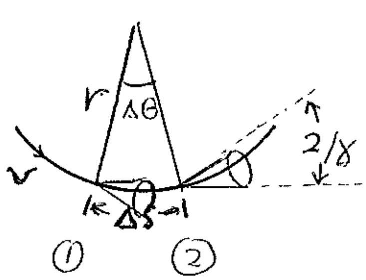


S frame



S' frame (rest)

The electric field seen by the observer lasts for a short time compared to the gyration period. During this time the electron travels a distance $\Delta S = r\,\Delta\theta = \frac{2}{\gamma} r$, where r is the radius of curvature, not the gyroradius. To find r, note that



$$\left.\begin{array}{l} a = \omega_B v_\perp \\ \frac{v^2}{r} = \omega_B v \sin\alpha \end{array}\right\} \Rightarrow r = \frac{v}{\omega_B \sin\alpha}$$

So $\Delta S = \frac{2v}{\gamma\omega_B \sin\alpha}$

The time it takes for the electron to travel from point ① to point ② is $t_2 - t_1 = \Delta S / v$

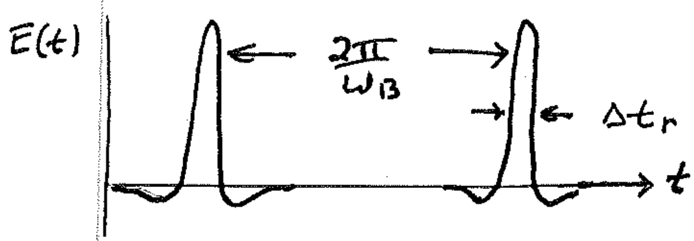
$$t_2 - t_1 = \frac{2}{\gamma\omega_B \sin\alpha}$$

The time it takes for the pulse of radiation to arrive is shorter, $\Delta t_r = t_2 - t_1 - \frac{\Delta S}{c}$

$$\Delta t_r = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right)$$

Approximate $1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$

$$\Delta t_r \approx \frac{1}{\gamma^3 \omega_B \sin\alpha} \ll \frac{1}{\omega_B}$$



The spectrum of the radiation will contain frequencies up to $\approx 1/\Delta t_r$, the width of the pulse

Define a "critical frequency" $\omega_c \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha$.

We will see that the spectrum of synchrotron radiation from a single electron is

$$P(\omega) = \frac{\sqrt{3}}{2\pi}\frac{e^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right) \quad \left[\frac{\text{erg}}{\text{s Hz}}\right]$$