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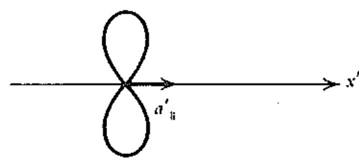
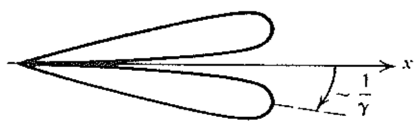
Dipole Radiation from Relativistic Particle



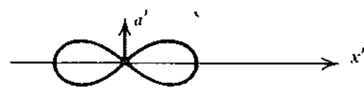
(a)

***Figure 4.11a** Dipole radiation pattern for particle at rest.*



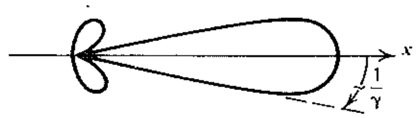
(b)

***Figure 4.11b** Angular distribution of radiation emitted by a particle with parallel acceleration and velocity.*



(c)

***Figure 4.11c** Same as a.*



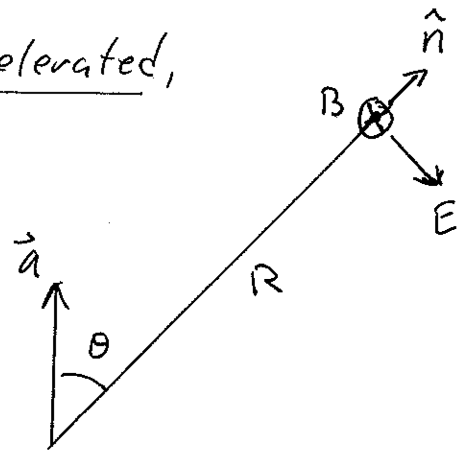
(d)

***Figure 4.11d** Angular distribution of radiation emitted by a particle with perpendicular acceleration and velocity.*

3—Extreme Relativistic Limit. When $\gamma \gg 1$, the quantity $(1-\beta\mu)$ in the denominators becomes small in the forward direction, and the radiation becomes strongly peaked in this direction. Using the same arguments as before, we obtain

$$(1-\beta\mu) \approx \frac{1+\gamma^2\theta^2}{2\gamma^2}.$$

Review of radiation from an accelerated, non-relativistic charge



$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \quad \text{where}$$

$$\vec{E} = \frac{q}{c^2 R} \hat{n} \times (\hat{n} \times \vec{a})$$

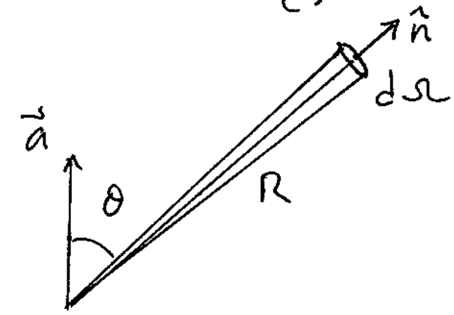
$$\vec{B} = \hat{n} \times \vec{E}$$

$$|E| = |B| = \frac{q}{c^2 R} a \sin\theta$$

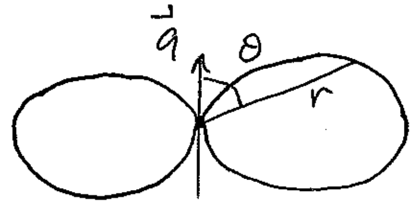
$$S = \frac{q^2}{4\pi c^3 R^2} a^2 \sin^2\theta \quad \left[\frac{\text{erg}}{\text{cm}^2\,\text{s}}\right]$$

$$\text{Power } P = \int S \, dA = \int S R^2 d\Omega = \frac{2}{3} \frac{q^2}{c^3} a^2$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\theta}$$

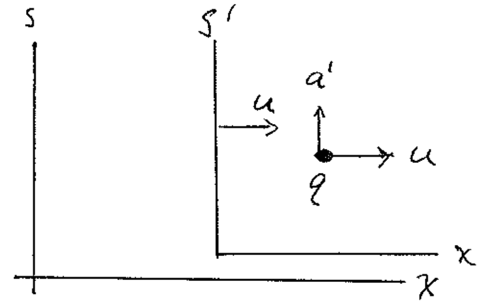


Angular distribution of power is a torus with axis in the direction of $\vec{a}$



This is a polar coordinates diagram where the radius r is proportional to $\sin^2\theta$. It is independent of the direction of the velocity of the particle as long as it is non-relativistic ($\beta \ll 1$).

Larmor's formula is only valid for $\beta \ll 1$, e.g., in the rest (S') frame of the charge. (You can consider an instantaneous rest frame). $u' = 0$ in this frame.



Power $\mathbb{P}' = \frac{2}{3} \frac{q^2}{c^3} (a')^2$

a' is the acceleration in frame S'!

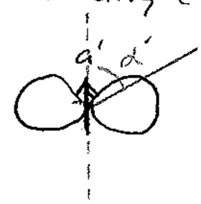
Consider the radiation from the charge to be the sum of all the 4-momentum of the radiation

$\Sigma p' = (E'/c , 0)$ in frame S'

$\Sigma p = (E/c , \vec{p})$ in frame S

The 3-momentum of the radiation in S' is zero because dipole radiation is symmetric with respect to any direction:

$$\frac{d\mathbb{P}'}{d\Omega'} = \frac{q^2}{4\pi c^3} (a')^2 \sin^2\alpha'$$



The total power emitted is $\frac{dE'}{dt'}$ or $\frac{dE}{dt}$.

According to Lorentz transformation

$$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'/c \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} E/c = \gamma E'/c \\ \text{or } dE = \gamma dE' \end{matrix}$$

Since $dt = \gamma dt'$ by same Lorentz transformation $\frac{dE}{dt} = \frac{dE'}{dt'}$.

The total power emitted is a Lorentz invariant because the emission is symmetric forward and backward in the instantaneous rest frame

$$\boxed{\mathbb{P} = \mathbb{P}'}$$

$$\Rightarrow \mathbb{P} = \frac{2q^2}{3c^3} \vec{a}' \cdot \vec{a}' = \frac{2q^2}{3c^3} \left[(a'_\parallel)^2 + (a'_\perp)^2 \right]$$

Lorentz transformation of acceleration are

$$a'_{\parallel} = \frac{a_{\parallel}}{\gamma^3 (1 - \vec{\beta}\cdot\vec{u}/c)^3} \qquad a'_{\perp} = \frac{a_{\perp} - \vec{\beta}\times(\vec{a}\times\vec{u})/c}{\gamma^2 (1-\vec{\beta}\cdot\vec{u}/c)^3}$$

where parallel and perpendicular refer to the direction between $\vec{a}$ and $\vec{u}$. Note that if $\vec{\beta} = \vec{u}/c$ the particle's rest frame is the moving frame S', and

$$\boxed{a'_{\parallel} = \gamma^3 a_{\parallel}} \qquad \boxed{a'_{\perp} = \gamma^2 a_{\perp}}$$

Therefore, the total power in any frame is

$$\mathbb{P} = \mathbb{P}' = \frac{2q^2}{3c^3}\left[(a'_{\parallel})^2 + (a'_{\perp})^2\right] = \frac{2q^2}{3c^3}\gamma^4 (a_{\perp}^2 + \gamma^2 a_{\parallel}^2)$$

Now we would like to know the <u>angular distribution</u> of the power:

$$\frac{d\mathbb{P}}{d\Omega} = \frac{dE}{dt\, d\Omega}$$

Since we know $\frac{dE'}{dt'd\Omega'}$ in the rest frame of the particle, use the Lorentz transformation for each of the three primed quantities.

$$E = \gamma(E' + v p'_x)$$
$$dE = \gamma(dE' + v\, dp'_x)$$

For a photon $p' = E'/c$

$$p'_x = \cos\theta' E'/c$$

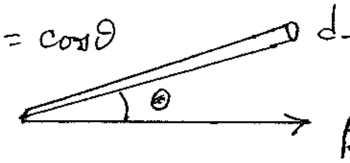
$$dE = \gamma\, dE'(1 + \beta\cos\theta')$$

$$\boxed{dE = \gamma(1+\beta\mu')\, dE'}$$

Time interval is

$$\boxed{dt = \gamma\, dt'}$$

$\mu = \cos\theta$ $\qquad d\Omega = d\mu\, d\varphi$



From aberration formula

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \qquad \varphi = \varphi'$$

$$d\mu = \frac{[1 + \beta\mu' - \beta(\mu' + \beta)]\, d\mu'}{(1+\beta\mu')^2}$$

$$\boxed{d\Omega = \frac{d\Omega'}{\gamma^2 (1+\beta\mu')^2}}$$

The angular distribution of the power emitted in the lab frame is then

$$\frac{d\mathbb{P}_e}{d\Omega} = \frac{dE}{dt\,d\Omega} = \gamma^2(1+\beta\mu')^3 \frac{d\mathbb{P}'}{d\Omega'}$$

However the time interval during which the emitted power is received by an observer in the lab frame S depends on the location of the observer relative to the direction of β, similar to the Doppler shift. The time interval of received power is dt_r, where

$dt_r = (1-\beta\cos\theta)\,dt$

use time dilation $\Rightarrow dt_r = \gamma(1-\beta\cos\theta)\,dt'$

Then exchange prime and unprime, change sign β

$\Rightarrow dt' = \gamma(1+\beta\cos\theta')\,dt_r$

[Remember Doppler shift:
$\nu = \dfrac{\nu'}{\gamma(1-\beta\cos\theta)}$
or $\nu = \nu'(1+\beta\cos\theta')\gamma$]

$$1+\beta\mu' = \frac{dt'}{\gamma\,dt_r} = \frac{1}{\gamma^2(1-\beta\mu)}$$

Substitute this into the expression for $\frac{d\mathbb{P}_e}{d\Omega}$ above.

$$\frac{d\mathbb{P}_e}{d\Omega} = \frac{1}{\gamma^4}\frac{1}{(1-\beta\mu)^3}\frac{d\mathbb{P}'}{d\Omega'}$$

This is the emitted power.

The angular distribution of the received power is

$$\boxed{\frac{d\mathbb{P}_r}{d\Omega} = \frac{d\mathbb{P}_e}{d\Omega}\frac{dt}{dt_r} = \frac{1}{\gamma^4}\frac{1}{(1-\beta\mu)^4}\frac{d\mathbb{P}'}{d\Omega'}}$$

because $dt_r = (1-\beta\cos\theta)\,dt$

The angular distribution of power has a maximum at $\theta = 0$ $(\mu = 1)$, and becomes more highly peaked in this direction as $\beta \to 1$.