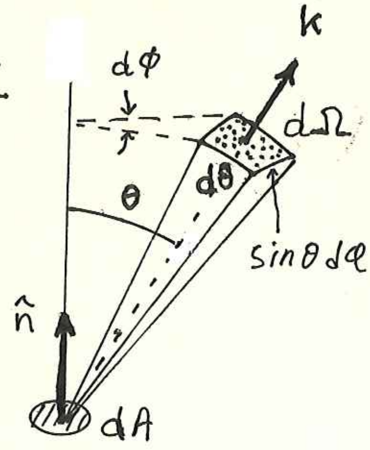
Specific Intensity I_ν and its Moments

Define specific intensity I_ν in terms of the energy per unit area, per unit time, per unit solid angle, per unit frequency going in direction of unit vector $\hat{k}$ at an angle θ to the unit vector $\hat{n}$ that is normal to the surface area element dA



$d\Omega = \sin\theta\, d\theta\, d\varphi$
(solid angle)

$$\boxed{dE = I_\nu\, dA\, dt\, d\Omega\, d\nu\ \hat{k}\cdot\hat{n}}$$

$I_\nu(\hat{k}) = I_\nu(\theta, \phi)$ has units $[\text{erg cm}^{-2}\, \text{s}^{-1}\, \text{ster}^{-1}\, \text{Hz}^{-1}]$

Specific flux is the 1^{st} moment of I_ν

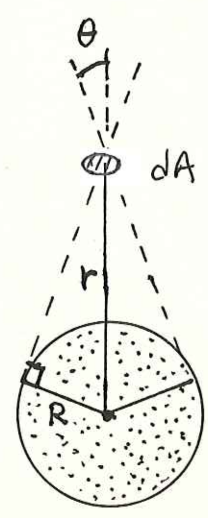
$$\boxed{F_\nu = \int I_\nu \cos\theta\, d\Omega} \quad \left[\frac{\text{erg}}{\text{cm}^2\, \text{s Hz}}\right] \qquad \cos\theta = \hat{k}\cdot\hat{n}$$

Example: an isotropically emitting surface (I_ν is independent of θ and ϕ). What is F_ν at the surface?

$$F_\nu = I_\nu \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi = I_\nu \int_0^1 x\, dx \cdot 2\pi$$

$$F_\nu = \pi I_\nu$$

Example: an isotropically emitting star of radius R. What is the specific flux at a distance r from the star?



$$F_\nu = I_\nu \int_0^{\sin^{-1}(R/r)} \cos\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$F_\nu = 2\pi I_\nu \int_0^{R/r} x\, dx = \pi I_\nu \left(\frac{R}{r}\right)^2$$

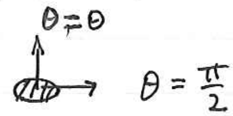
The flux decreases as the inverse square of the distance, while the specific intensity is constant along a ray (in a vacuum). I_ν is a "surface brightness" in these examples, which is independent of distance. Also, when intensity is isotropic, the surface brightness is uniform.

<u>Mean intensity is the 0^{th} moment of I_ν</u>

$$\boxed{J_\nu = \frac{1}{4\pi}\int I_\nu \, d\Omega}$$ has the same units as I_ν

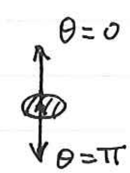
Example: at the surface of a uniformly bright object

$$J_\nu = \frac{I_\nu}{4\pi}\int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{I_\nu}{2}$$



Example: <u>inside</u> an isotropic radiation field

$$J_\nu = \frac{I_\nu}{4\pi}\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = I_\nu$$



but $$F_\nu = \int I_\nu \cos\theta \, d\Omega = I_\nu \int_0^{\pi} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi = 0$$

The flux inside an <u>isotropic</u> radiation field is zero, while the mean intensity $J_\nu = I_\nu$.

<u>Specific radiation pressure is the 2^{nd} moment of I_ν</u>

$$\boxed{P_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega} \quad \left[\frac{\text{erg}}{\text{cm}^3 \text{ Hz}}\right] \quad \left[\frac{\text{dynes}}{\text{cm}^2 \text{ Hz}}\right]$$

Radiation pressure has the same units as energy density, and also as momentum flux, as you can see by writing them as $\left[\frac{\text{dynes} \cdot \text{s}}{\text{cm}^2 \text{ s Hz}}\right]$ (Momentum is dynes $\cdot$ s).

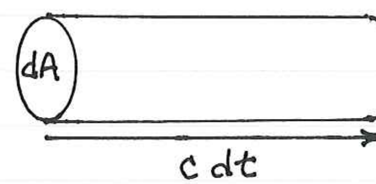
But are energy density, radiation pressure, and momentum flux the same physical quantities? Energy is a scalar, while momentum is a vector, so we need the extra factor of $\cos\theta$ to specify the normal component of momentum. So radiation pressure = momentum flux, but energy density is a different quantity.

Energy density per unit solid angle

$u_\nu(\Omega)$ is the specific energy density per unit solid angle.

The energy in a cylinder moving in the direction of the arrow is

$$dE = u_\nu(\Omega)\, \underbrace{dA\, c\, dt}_{\text{volume}}\, d\Omega\, d\nu$$



But we also define I_ν using
$dE = I_\nu\, dA\, dt\, d\Omega\, d\nu$

So $u_\nu(\Omega) = \frac{I_\nu}{c}$

Integrating over solid angle, $u_\nu = \int u_\nu(\Omega)\, d\Omega$ gives

$$\boxed{u_\nu = \frac{4\pi J_\nu}{c}}$$

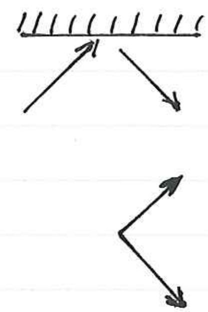
(recall $J_\nu = \frac{1}{4\pi}\int I_\nu\, d\Omega$)

So energy density is another form of the 0^{th} moment.

Pressure = Momentum flux

Because of the vector property of momentum, pressure is the same on the walls of a container of isotropic radiation as it is in the interior.

Consider the z-component of momentum. Rays moving in the +z direction carry positive momentum in the +z direction, while rays traveling in the −z direction carry negative momentum in the −z direction. They both contribute to positive momentum flux.



Unlike $F_\nu = 0$ in an isotropic radiation field,

$$P_\nu = \frac{I_\nu}{c} \int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi = 2\pi \int_{-1}^{1} \mu^2 d\mu = \frac{4\pi}{3c} I_\nu$$

We showed that in an isotropic radiation field $u_\nu = \frac{4\pi J_\nu}{c}$, where $J_\nu = I_\nu$. Therefore,

$$P_\nu = \frac{u_\nu}{3} \quad \text{in an isotropic radiation field.}$$

<u>Moments of I_ν as tensors</u>

We have been treating all the moments as scalar quantities, but a more general approach treats flux as a vector and pressure as a tensor.

0^{th}	$u_\nu = \frac{1}{c}\int I_\nu \, d\Omega$	scalar energy density
1^{st}	$\vec{F}_\nu = \int I_\nu \hat{k} \, d\Omega$	vector flux
2^{nd}	$\overleftrightarrow{P}_\nu = \frac{1}{c}\int I_\nu \hat{k}\hat{k} \, d\Omega$	pressure tensor

Note that $\vec{F}_\nu \cdot \hat{n} = \int I_\nu \hat{k}\cdot\hat{n} \, d\Omega = \int I_\nu \cos\theta \, d\Omega$ is the scalar flux. If radiation is absorbed, the flux vector can be related to the force per unit volume of the radiation acting on the matter, $\vec{\mathcal{F}}$.

$$\vec{\mathcal{F}} = \int \frac{\alpha_\nu}{c} \vec{F}_\nu \, d\nu \qquad \left[\frac{\text{force}}{\text{vol}}\right]$$

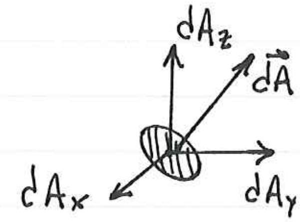
where α_ν is the absorption coefficient [cm^{-1}], to be defined later.

The components of the pressure tensor are

$$P_{ij} = \frac{1}{c} \int I_\nu k_i k_j \, d\Omega = \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix}$$

The component P_{ij} is the flux of i-momentum in the j-direction. The vector flux across surface area $d\vec{A}$ is the dot product of $\overleftrightarrow{P}$ and $d\vec{A}$

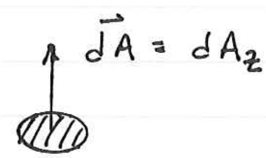
$$d\vec{F}_\nu = \overleftrightarrow{P}_\nu \cdot d\vec{A} \quad , \quad \text{or } dF_i = \sum_j P_{ij} \, dA_j$$



e.g., $dF_x = P_{xx} \, dA_x + P_{xy} \, dA_y + P_{xz} \, dA_z$

Even if $dA_x = dA_y = 0$, there can in principle be three components of momentum flux and three components of force:

$$dF_x = P_{xz} \, dA_z$$
$$dF_y = P_{yz} \, dA_z$$
$$dF_z = P_{zz} \, dA_z$$



The pressure tensor is symmetric: $P_{ij} = P_{ji}$

The energy density u is the trace of $\overleftrightarrow{P}$ (the sum of the diagonal elements):

$$u = P_{xx} + P_{yy} + P_{zz}$$

If the radiation is isotropic the scalar pressure P can be used, where

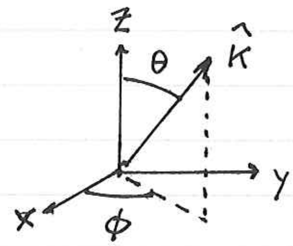
$$P = P_{xx} = P_{yy} = P_{zz} = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})$$

and $P = \frac{u}{3}$ if the radiation is isotropic.

Evaluating the components of $\overleftrightarrow{P}$

Write the components of the unit vector $\hat{k}$ in cartesian coordinates:

$$k_x = \sin\theta\cos\phi$$
$$k_y = \sin\theta\sin\phi$$
$$k_z = \cos\theta$$



Example: I_ν isotropic

$$P_{xx} = \frac{I_\nu}{c}\int_0^\pi k_x k_x \sin\theta\, d\theta \int_0^{2\pi} d\phi$$ let $\mu = \cos\theta$

$$= \frac{I_\nu}{c}\int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi}\cos^2\phi\, d\phi = \frac{\pi I_\nu}{c}\int_{-1}^{1}(1-\mu^2)\, d\mu$$

$$= \frac{4\pi I_\nu}{3c}$$

$$P_{yy} = \frac{I_\nu}{c}\int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi}\sin^2\phi\, d\phi = \frac{4\pi}{3}\frac{I_\nu}{c}$$

$$P_{zz} = \frac{I_\nu}{c}\int_0^\pi \cos^2\theta\sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{2\pi I_\nu}{c}\int_{-1}^{1}\mu^2\, d\mu$$

$$= \frac{4\pi}{3}\frac{I_\nu}{c}$$

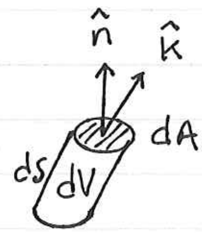
$$P_{xy} = \frac{I_\nu}{c}\int_0^\pi \sin^3\theta\, d\theta \underbrace{\int_0^{2\pi}\cos\phi\sin\phi\, d\phi}_{0} = 0, \text{ etc.}$$

All off-diagonal terms are zero.

So far we have been talking about radiation in a vacuum. Radiative transfer is about the interaction of radiation with matter, through emission and absorption.

Radiative Transfer - Emission

The amount of energy added to a ray, per unit time, solid angle, frequency, from a volume element dV, is



$$dE = j_\nu \, dV \, d\Omega \, dt \, d\nu$$

where j_ν is the specific (monochromatic) emission coefficient

$$j_\nu \left[\frac{\text{erg}}{\text{cm}^3 \text{ s ster Hz}}\right]$$

Since $dV = dA \, ds \, \hat{k}\cdot\hat{n}$, we can write

$$dE = j_\nu \, dA \, ds \, d\Omega \, dt \, d\nu \, \hat{k}\cdot\hat{n}$$

From the definition of specific intensity, the increase in I_ν from the added energy dE is dI_ν, where

$$dE = dI_\nu \, dA \, d\Omega \, dt \, d\nu \, \hat{k}\cdot\hat{n}$$

Therefore $dI_\nu = j_\nu \, ds$, or $\frac{dI_\nu}{ds} = j_\nu$

Note: j_ν can be a function of direction, but if it is isotropic, we sometimes use the volume emission coefficient, which is integrated over solid angle.

$$\epsilon_\nu = \int j_\nu \, d\Omega \quad (= 4\pi j_\nu \text{ if isotropic})$$

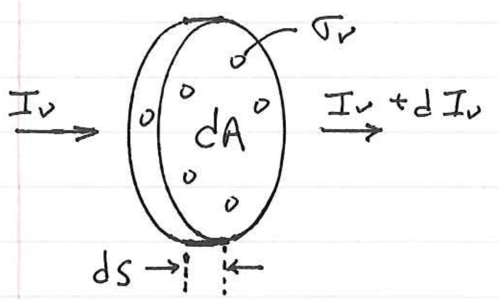
ϵ_ν has units $\left[\frac{\text{erg}}{\text{cm}^3\ \text{s Hz}}\right]$.

Radiative Transfer - Absorption

An absorption coefficient α_ν [cm^{-1}] can be expressed in terms of the microscopic cross-sectional area of an atom, σ_ν [cm^2] and the number density of atoms, n [cm^{-3}], as

$$\alpha_\nu = n\,\sigma_\nu$$

To see this, consider a thin slab of absorbing material such that the probability of a ray encountering an absorber is $\ll 1$. The volume of the slab is $dV = dA\,ds$.



The number of absorbers in the slab is $n\,dA\,ds$. The total area of the absorbers is $n\,dA\,ds\,\sigma_\nu$. The relative change in intensity of a ray passing through the slab is:

$$\frac{dI_\nu}{I_\nu} = -\frac{n\,dA\,ds\,\sigma_\nu}{dA},$$

which is the fraction of the area of the slab subtended by the absorbers. Therefore

$$\frac{dI_\nu}{ds} = -n\,\sigma_\nu\,I_\nu = -\alpha_\nu\,I_\nu$$

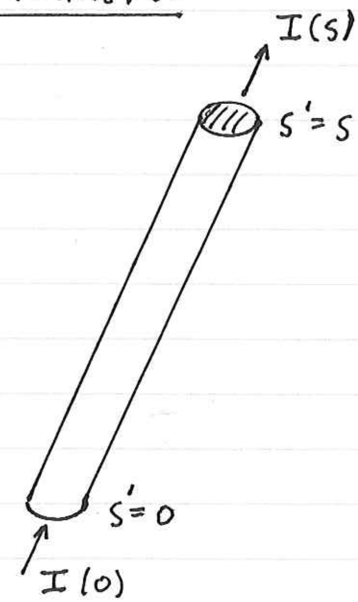
where $\alpha_\nu \equiv n\,\sigma_\nu$ [cm^{-1}]

Now including both emission and absorption processes, we have

$$\boxed{\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu}$$ The Equation of Radiative Transfer

This is not quite complete, because we have not yet considered an additional process, scattering, which will be included later.

Solutions of the Equation of Radiative Transfer



1. Emission only ($\alpha_\nu = 0$)

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')\,ds'$$

2. Absorption only ($j_\nu = 0$)

$$\int_0^s \frac{dI_\nu}{I_\nu} = -\int_0^s \alpha_\nu(s')\,ds'$$

$$\ln\left(\frac{I_\nu(s)}{I_\nu(0)}\right) = -\int_0^s \alpha_\nu(s')\,ds'$$

In the case α_ν is independent of position s, then

$$I_\nu(s) = I_\nu(0)\,e^{-\alpha_\nu s}$$

Definition of Optical Depth τ_ν

$d\tau_\nu = \alpha_\nu\,ds$, a dimensionless quantity

$$\tau_\nu(s) = \int_0^s \alpha_\nu(s')\,ds'$$

$\tau_\nu \gtrsim 1$ optically thick

$\tau_\nu \ll 1$ optically thin

$$\frac{dI_\nu}{\alpha_\nu ds} = \frac{j_\nu}{\alpha_\nu} - I_\nu$$

Change independent variable from s to τ_ν.

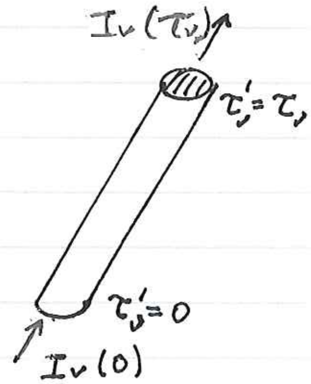
$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu}$$

where $\boxed{S_\nu \equiv \frac{j_\nu}{\alpha_\nu}}$ "Source Function"

Formal Solution of the Equation of Radiative Transfer

Multiply the Equation by e^{τ_ν}, an integrating factor

$$\frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} + I_\nu e^{\tau_\nu} = S_\nu e^{\tau_\nu}$$



$$\frac{d}{d\tau_\nu}\left[I_\nu e^{\tau_\nu}\right] = S_\nu e^{\tau_\nu}$$

$$\int_0^{\tau_\nu} d\left[I_\nu e^{\tau_\nu'}\right] = \int_0^{\tau_\nu} S_\nu e^{\tau_\nu'} d\tau_\nu'$$

$$e^{\tau_\nu} I_\nu(\tau_\nu) - I_\nu(0) = \int_0^{\tau_\nu} S_\nu e^{\tau_\nu'} d\tau_\nu'$$

$$\boxed{I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu') \, d\tau_\nu'}$$

Example: A uniform source function S_ν (independent of position).

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu e^{-\tau_\nu} \int_0^{\tau_\nu} e^{\tau_\nu'} d\tau_\nu'$$

$$= I_\nu(0) e^{-\tau_\nu} + S_\nu e^{-\tau_\nu} (e^{\tau_\nu} - 1)$$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$$

Limiting cases of this solution are informative:
If $\tau_\nu \ll 1$, approximate $e^{-\tau_\nu} \approx 1 - \tau_\nu$

1. If $\tau_\nu \ll 1$ $\quad I_\nu \to I_\nu(0) + S_\nu \tau_\nu = I_\nu(0) + j_\nu S$

2. If $\tau_\nu \gg 1$ $\quad I_\nu \to S_\nu$

"Specific Intensity approaches the source function in the limit of large optical depth." From the Equation of radiative transfer, if $I_\nu < S_\nu$, then I_ν increases along a ray, while if $I_\nu > S_\nu$, I_ν decreases along the ray.

<u>Mean-Free-Path</u> - average distance travelled before absorption

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu}$$

The probability of a photon travelling to optical depth τ_ν without being absorbed is just the ratio of the Intensities of the ray at τ_ν and 0.

$$\text{Prob}(\geq \tau_\nu) = \frac{I_\nu(\tau_\nu)}{I_\nu(0)} = e^{-\tau_\nu}$$

So $e^{-\tau_\nu}$ is a probability density function for the distance travelled by a photon. The average optical depth travelled is then

$$\langle \tau_\nu \rangle = \frac{\int_0^\infty \tau_\nu e^{-\tau_\nu}\, d\tau}{\int_0^\infty e^{-\tau_\nu}\, d\tau} = 1$$

The mean-free-path is the distance l_ν such that $\tau_\nu = 1$

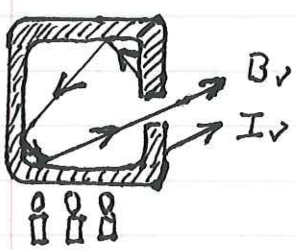
$$\tau_\nu = \alpha_\nu S$$
$$1 = \alpha_\nu S \quad \Rightarrow \quad l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}$$

Blackbody Radiation $B_\nu(T)$

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \quad \left[\frac{\text{erg}}{\text{cm}^2\,\text{s ster Hz}}\right]$$

- $B_\nu(T)$ is the Planck function, which is a specific intensity. It is isotropic and unpolarized.
- B_ν is independent of the properties of the enclosure, and depends only on temperature T.
- A black body is a perfect emitter and a perfect absorber of radiation.

Also known as "cavity radiation", its properties can be understood with the following thought experiment, consisting of a heated oven with a small hole.



If the oven is heated to a uniform temperature, a ray emitted from anywhere on the surface (inside or outside) will have the same I_ν. But rays hitting an inside surface will be reflected or absorbed with probability r and a, respectively, such that $a+r=1$. Then a ray emerging from the hole will have

$$B_\nu = I_\nu + I_\nu r + I_\nu r^2 + \ldots = \frac{I_\nu}{1-r} = \frac{I_\nu}{a}$$

Therefore, $I_\nu = B_\nu a$

If $a=1$ $\quad I_\nu = B_\nu$ $\quad$ "a perfect emitter is a perfect absorber"

If $a<1$ $\quad I_\nu < B_\nu$

For blackbody radiation, $I_\nu = B_\nu$, and I_ν is the maximum possible at all frequencies for thermal radiation at the given temperature T. The equation of radiative transfer becomes

$$\frac{dI_\nu}{d\tau} = -B_\nu + S_\nu$$

But I_ν cannot change if it is blackbody, so $S_\nu = B_\nu$, therefore $j_\nu = \alpha_\nu B_\nu(T)$. The emission and absorption coefficients for matter at temperature T are related by the Planck function.

Key Summary

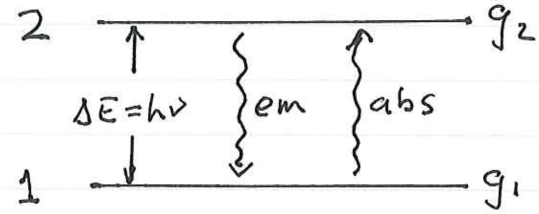
$S_\nu = B_\nu(T)$ for any thermal radiation
$I_\nu = B_\nu(T)$ for black body radiation only

$j_\nu = \alpha_\nu B_\nu(T)$ for matter in local thermodynamic equilibrium (LTE), Kirchoff's Law

$\frac{dI_\nu}{d\tau} = -I_\nu + B_\nu$ is the equation of radiative transfer for thermal radiation, which is emitted by matter in LTE

Black body radiation is thermal radiation that is emitted by an optically thick medium. Such a situation is known as thermal equilibrium (TE) between matter and radiation. In thermal equilibrium, detailed balance applies between any pair of atomic levels emitting and absorbing radiation. This can be used to derive the relations between the Einstein coefficients, which are the parts of α_ν and j_ν that depend on microscopic physics.

Einstein coefficients



Consider a two-level system emitting and absorbing photons of energy $\Delta E = h\nu$

A_{21} $[s^{-1}]$ spontaneous emission rate

$B_{12} J_\nu$ $[s^{-1}]$ absorption rate

$B_{21} J_\nu$ $[s^{-1}]$ stimulated emission rate

In thermal equilibrium (TE) detailed balance applies

$$n_1 B_{12} J_\nu = n_2 A_{21} + n_2 B_{21} J_\nu$$

where n_1 and n_2 are the densities of "atoms" in levels 1 and 2. Solving for J_ν,

$$J_\nu = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$$

But in local thermodynamic equilibrium (LTE), where energy levels are populated according to temperature T:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT},$$

where g_1 and g_2 are the statistical weights, equivalent to the number of available states in each energy level.

Therefore
$$J_\nu = \frac{A_{21}/B_{21}}{\left(\frac{g_1}{g_2}\frac{B_{12}}{B_{21}}\right)e^{h\nu/kT} - 1}$$

In TE (radiation in equilibrium with matter) $J_\nu = B_\nu$. Therefore, it is required that

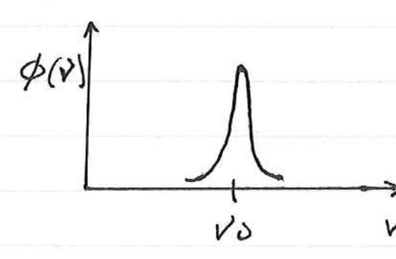
$$\frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2} \quad \text{and} \quad g_1 B_{12} = g_2 B_{21}$$

Stimulated emission with rate B_{21} was required to explain the Planck function. Note that the Einstein coefficients have no dependence on T, but are properties of the microphysics of the "atom". It is sufficient to know any one of A_{21}, B_{21}, or B_{12}, to know them all.

Line Profile Function $\phi(\nu)$

Since energy levels are not infinitesimally precise, a line profile function must be used to describe the transition rates as a function of frequency

$$\int_0^\infty \phi(\nu)\, d\nu = 1$$



Instead of mean intensity J_ν, write
$$J = \int_0^\infty J_\nu\, \phi(\nu)\, d\nu$$

Now relate the Einstein coefficients to the emission and absorption coefficients j_ν and α_ν in the equation of radiative transfer.

For emission $dE = j_\nu \, dV \, d\Omega \, dt \, d\nu$

$j_\nu \left[\frac{\text{erg}}{\text{cm}^3 \text{ ster s Hz}} \right]$

$$\boxed{j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)} \qquad \left[\frac{\text{ergs}}{\text{ster}} \text{cm}^{-3} \text{s}^{-1} \text{Hz}^{-1} \right]$$

For absorption $dI_\nu = -\alpha_\nu I_\nu \, ds$

where $dI_\nu = \frac{dE}{dA \, d\Omega \, dt \, d\nu}$, therefore

$$\alpha_\nu = \frac{-dE}{dA \, ds \, d\Omega \, dt \, d\nu \, I_\nu} \qquad \left[\frac{\text{erg}}{\text{cm}^3 \text{ ster s Hz}} \right] \left[\frac{1}{I_\nu} \right]$$

$$\boxed{\alpha_\nu = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu)}$$ absorption uncorrected for stimulated emission

or $$\boxed{\alpha_\nu = \frac{h\nu_0}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu)}$$ corrected for stimulated emission

Stimulated emission is formally equivalent to negative absorption, because it is proportional to the intensity of radiation, and it only affects the intensity along the ray.

The source function can now be seen to depend only on the level populations because the Einstein coefficients drop out.

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{n_2 B_{21}}{(n_1 B_{12} - n_2 B_{21})} \frac{2h\nu^3}{c^2}$$

$$S_\nu = \frac{1}{\left(\frac{n_1 B_{12}}{n_2 B_{21}} - 1\right)} \frac{2h\nu^3}{c^2} = \frac{1}{\left(\frac{n_1 g_2}{n_2 g_1} - 1\right)} \frac{2h\nu^3}{c^2}$$

$$S_\nu = \left(\frac{1}{e^{h\nu/kT} - 1}\right) \frac{2h\nu^3}{c^2}$$ because the level populations are in LTE

Non-thermal emission occurs when LTE is not satisfied. A special case of non-thermal emission occurs when level populations are "inverted."

In normal populations $\frac{n_2}{n_1} < \frac{g_2}{g_1}$

But if $\frac{n_2}{n_1} > \frac{g_2}{g_1}$ then the absorption

coefficient (corrected for stimulated emission) is negative, and the intensity increases along a ray. This is equivalent to negative optical depth in the equation of radiative transfer, and is "light amplification by stimulated emission of radiation" (LASER), or MASER.

<u>Some useful integrals of B_ν</u>

Surface flux of an isotropic sources is $F_\nu = \pi I_\nu$, therefore

$$F_\nu = \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$ for a B.B.

- Total flux $F = \int_0^\infty F_\nu \, d\nu = \sigma T^4 \quad \left[\frac{\text{erg}}{\text{cm}^2\,\text{s}}\right]$

where $\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \frac{\text{erg}}{\text{cm}^2 \text{ s K}^4}$

- Energy density $u_\nu = \frac{4\pi J_\nu}{c}$ where $J_\nu = B_\nu$

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

Total energy density $u = \int_0^\infty u_\nu \, d\nu = \frac{4\sigma}{c} T^4$

$$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15} \frac{\text{erg}}{\text{cm}^3 \text{ K}^4}$$

- Radiation pressure $P = \int P_\nu \, d\nu = \frac{1}{3} u = \frac{a T^4}{3} = \frac{4\sigma T^4}{3c}$

Rayleigh - Jeans Law

In the limit $h\nu \ll kT$, $e^{h\nu/kT} \approx 1 + \frac{h\nu}{kT}$

$$B_\nu = \frac{2\nu^2}{c^2} kT$$

Brightness temperature T_B

T_B is usually defined from the Rayleigh - Jeans law as the temperature of a black body that has the same brightness as any source of intensity I_ν. That is

$$I_\nu = \frac{2\nu^2}{c^2} k T_B$$

or $T_B = \frac{1}{2}\left(\frac{c}{\nu}\right)^2 \frac{I_\nu}{k}$, in general a function of frequency.