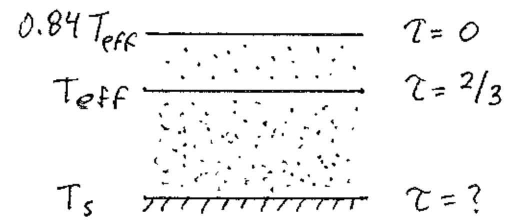
Earth's Greenhouse Effect - A Simple Model

Approximate the opacity in the infrared as grey. Estimate the optical depth and what it implies for the surface temperature, which is higher than the effective temperature implied by absorption and reemission of solar radiation. Average over the surface.



$$T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + 2/3\right)$$

$$f = \sigma T_{eff}^4 \quad \text{absorbed flux}$$

$$f = \frac{L_\odot}{4\pi r^2}(1-A)$$

where $L_\odot = 3.83 \times 10^{33}$ ergs s^{-1}
distance $r \approx 1$ AU $= 1.5 \times 10^{13}$ cm
albedo $A \approx 0.3$

$$\frac{L_\odot}{4\pi r^2}(1-A)\,\pi R_\oplus^2 = 4\pi R_\oplus^2 \sigma T_{eff}^4 \qquad \Rightarrow T_{eff} = 254\ K$$

The average surface temperature $T_s = 290$ K (17°C, 63°F).

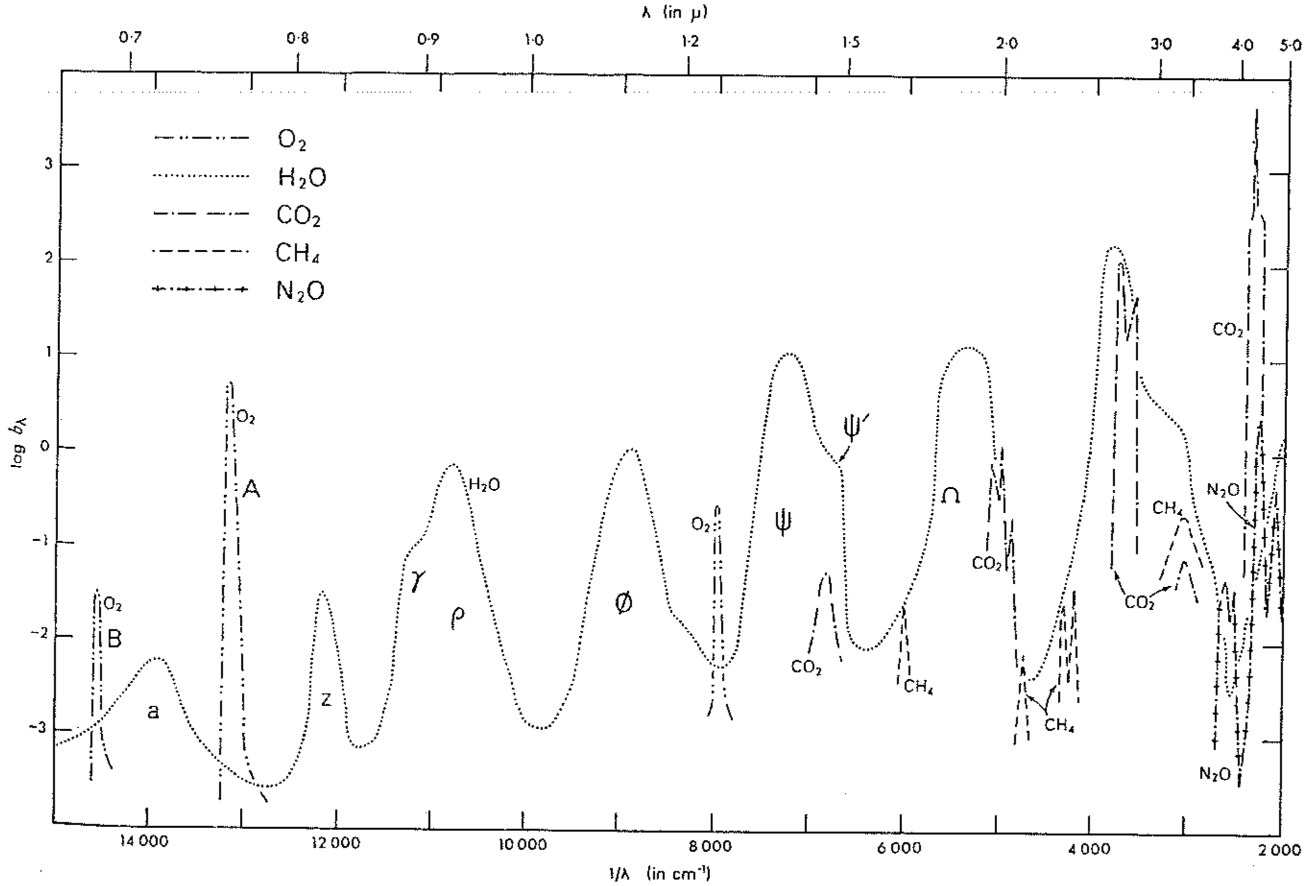
Use the grey opacity law to estimate the optical depth to the surface:

$$\tau \approx \frac{4}{3}\left(\frac{T_s}{T_{eff}}\right)^4 - \frac{2}{3} = \frac{4}{3}\left(\frac{290}{254}\right)^4 - \frac{2}{3} = 1.6$$

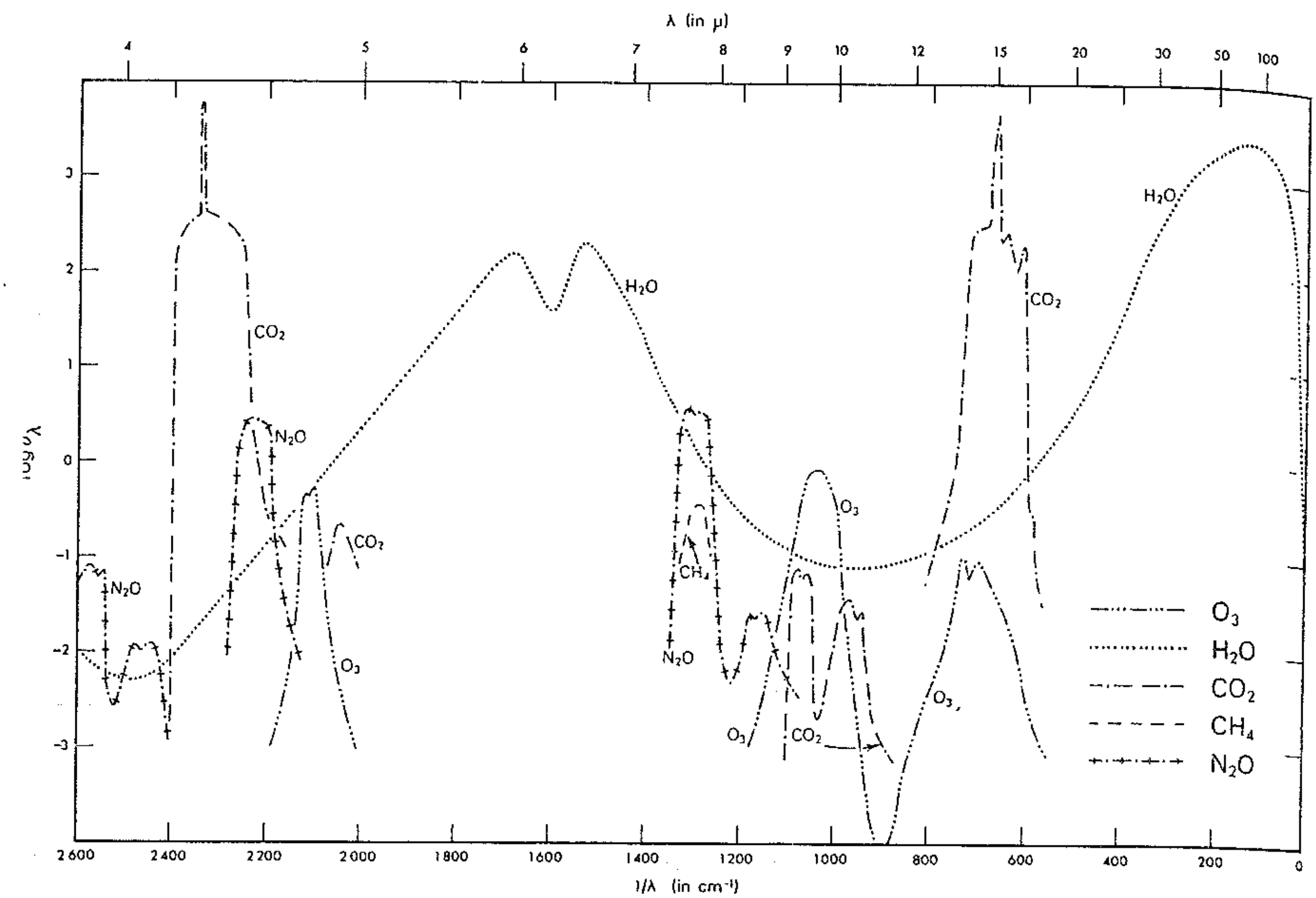
How sensitive is the surface temperature to the optical depth? Take the derivative $dT_s/d\tau$.

$$\frac{dT_s}{d\tau} = \frac{3}{16}\frac{T_{eff}^4}{T_s^3} = \frac{3}{16}\frac{254^4}{290^3} = 32$$

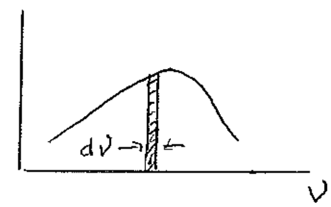
For example, if $d\tau = 0.1$, $dT_s = 3.^\circ 2$



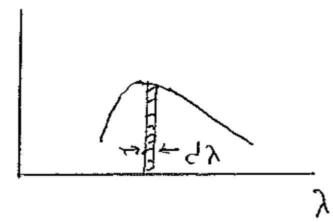
Infrared band absorption of atmospheric gases



$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}\left[\frac{\text{erg}}{\text{cm}^2\,\text{s Hz ster}}\right]$$



$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\left[\frac{\text{erg}}{\text{cm}^2\,\text{s cm ster}}\right]$$



$$B_\nu d\nu = B_\lambda d\lambda$$

The peak of B_ν is at $h\nu_{max} = 2.82\,kT$

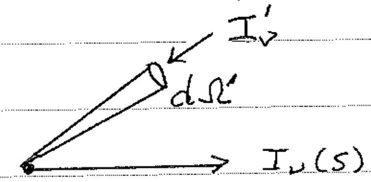
The peak of B_λ is at $\lambda_{max}T = 0.29$ cm K

For $T = 290$ K $\nu_{max} = 1.7\times10^{13}$ Hz $= 1.8\ \mu m$

but $\lambda_{max} = 10\ \mu m$

Scattering

The "emission" coefficient for coherent scattering (no change in energy of photon) is



$\left[\frac{\text{erg}}{\text{cm}^3\ \text{s}\ \text{sr}\ \text{Hz}}\right]$
$$j_\nu = \int \frac{d\sigma_\nu}{d\Omega'} I'_\nu(\Omega')\, d\Omega'$$

$[\text{cm}^{-1}\,\text{sr}^{-1}]$ where $\frac{d\sigma_\nu}{d\Omega}$ is the differential scattering coefficient.
The equation of radiative transfer for pure scattering is

$$\frac{dI_\nu}{ds} = -\sigma_\nu I_\nu + \int \frac{d\sigma_\nu}{d\Omega'} I'_\nu(\Omega')\, d\Omega'$$
integro-differential equation

$[\text{cm}^{-1}]$ where $\sigma_\nu = \int \frac{d\sigma_\nu}{d\Omega'} d\Omega'$ is the total scattering coefficient.
If the scattering process is <u>isotropic</u>, this simplifies:

$$\frac{dI_\nu}{ds} = -\sigma_\nu (I_\nu - J_\nu) \qquad \text{where } J_\nu = \frac{1}{4\pi}\int I_\nu\, d\Omega$$

In general, when there is scattering and absorption

$$\frac{dI_\nu}{ds} = -\sigma_\nu (I_\nu - J_\nu) - \alpha_\nu I_\nu + j_\nu$$
or
$$\frac{dI_\nu}{ds} = -\sigma_\nu (I_\nu - J_\nu) - \alpha_\nu (I_\nu - B_\nu)$$
when in the case of thermal emission, $j_\nu = \alpha_\nu B_\nu$.
We can derive an "average" source function S_ν by letting

$$\sigma_\nu J_\nu + \alpha_\nu B_\nu = (\sigma_\nu + \alpha_\nu) S_\nu$$

The source function is the average of scattering and absorption S_ν
Then the equation of radiative transfer becomes

$$\boxed{\frac{dI_\nu}{ds} = -(\sigma_\nu + \alpha_\nu)(I_\nu - S_\nu)}$$

Define optical depth $d\tau_\nu = (\sigma_\nu + \alpha_\nu)\,ds$, where the sum of the absorption and scattering coefficients is called the "extinction coefficient".

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu} \quad \text{where } S_\nu = \frac{\sigma_\nu J_\nu + \alpha_\nu B_\nu}{\sigma_\nu + \alpha_\nu}$$

The mean-free-path for either absorption or scattering is l_ν, where $l_\nu = \dfrac{1}{\sigma_\nu + \alpha_\nu}$. [cm]

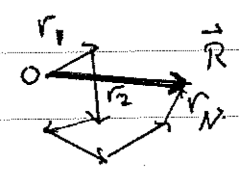
The probability of absorption in one mean-free-path is $\dfrac{\alpha_\nu}{\sigma_\nu + \alpha_\nu}$.

__Random walks__ - including effects of scattering and absorption.

1. __Pure scattering (no absorption)__

The distance travelled by a photon after N scatterings is the vector sum of the individual steps

$$\vec{R} = \vec{r}_1 + \vec{r}_2 + \ldots \vec{r}_N$$



The mean square distance is

$$\vec{R}\cdot\vec{R} = (r_1^2 + r_2^2 + \ldots r_N^2$$ ← Each term is $\approx l^2$

$$+ \vec{r}_1\cdot\vec{r}_2 + \ldots \vec{r}_1\cdot\vec{r}_N$$ ← All terms average to zero for isotropic scattering

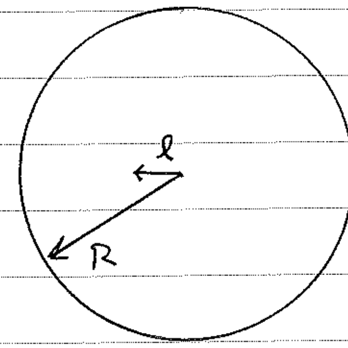
$$+ \ldots)$$

$$\langle R^2 \rangle = N l^2$$

$\sqrt{\langle R^2 \rangle} = \sqrt{N}\, l$ is the root-mean-square displacement

The path travelled by the photon is of length $N\ell$ on average, but it ends up a distance $\sqrt{N}\,\ell$ from its starting point.

Consider a photon starting from the center of a sphere of optical depth $\tau = R/\ell$.



If $\tau \ll 1$, the mean number of scatterings before escape is $1 - e^{-\tau} \approx \tau$. This is one minus the probability of not being scattered.

If $\tau \gg 1$, the mean number of scatterings before escape is $N = \langle R^2 \rangle / \ell^2 = \tau^2$.
In general, the mean number of scattering is

$$N \approx \max(\tau, \tau^2)$$

2. <u>Absorption and Scattering</u> m.f.p. $\ell = \frac{1}{\sigma_\nu + \alpha_\nu}$

$N = \frac{\sigma_\nu + \alpha_\nu}{\alpha_\nu}$ is the mean number of scatterings before absorption

$$\sqrt{\langle R^2 \rangle} = \sqrt{N}\,\ell = \sqrt{\frac{\sigma_\nu + \alpha_\nu}{\alpha_\nu}} \left(\frac{1}{\sigma_\nu + \alpha_\nu} \right) = \frac{1}{\sqrt{\alpha_\nu(\sigma_\nu + \alpha_\nu)}}$$

$\ell_* = \frac{1}{\sqrt{\alpha_\nu(\sigma_\nu + \alpha_\nu)}}$ is the "diffusion length" or "effective mean-free-path"

If $\alpha_\nu \ll \sigma_\nu$ then $\sqrt{\langle R^2 \rangle} = \frac{1}{\sqrt{\alpha_\nu \sigma_\nu}}$ (geometric mean)

If $\alpha_\nu \gg \sigma_\nu$ then $\sqrt{\langle R^2 \rangle} = \frac{1}{\alpha_\nu}$ (no scattering)

The effective optical thickness is the inverse of the diffusion length

$$\tau_* = \sqrt{\tau_a(\tau_a + \tau_s)}$$

where $\tau_a = \alpha_\nu R$, $\tau_s = \sigma_\nu R$

If $\tau_* \gg 1$ most photons will be destroyed before escaping, even if $\tau_a < 1$.

If $\tau_* \ll 1$ most photons will escape, but not necessarily in a straight line since it is possible that $\tau_s > 1$ (translucent).

Two-Stream Approximation

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - S_\nu$$ Eq. of radiative transfer

$$\frac{1}{4\pi}\frac{dF_\nu}{d\tau_\nu} = J_\nu - S_\nu$$ Zeroth moment equation

$$c\frac{dP_\nu}{d\tau_\nu} = F_\nu$$ First moment equation

$$c\frac{dP_\nu}{d\tau_\nu} = \frac{4\pi}{3}\frac{dJ_\nu}{d\tau_\nu}$$ From Eddington approximation where $P_\nu = \frac{u_\nu}{3} = \frac{4\pi J_\nu}{3c}$

$$\frac{1}{c}\frac{dF_\nu}{d\tau_\nu} = \frac{4\pi}{3c}\frac{d^2J_\nu}{d\tau_\nu^2} = \frac{4\pi}{c}(J_\nu - S_\nu)$$

$$\frac{1}{3}\frac{d^2J_\nu}{d\tau_\nu^2} = J_\nu - S_\nu$$

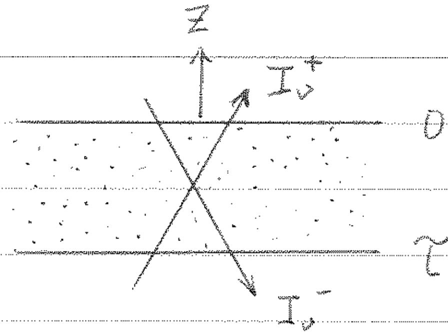
Since $S_\nu \equiv \dfrac{\sigma_\nu J_\nu + \alpha_\nu B_\nu}{\sigma_\nu + \alpha_\nu}$ we can write

$$\frac{1}{3}\frac{d^2J_\nu}{d\tau_\nu^2} = \left(1 - \frac{\sigma_\nu}{\sigma_\nu + \alpha_\nu}\right)J_\nu - \left(\frac{\alpha_\nu}{\sigma_\nu + \alpha_\nu}\right)B_\nu$$

$$\boxed{\frac{1}{3}\frac{d^2J_\nu}{d\tau_\nu^2} = \left(\frac{\alpha_\nu}{\sigma_\nu + \alpha_\nu}\right)(J_\nu - B_\nu)}$$ Radiative Diffusion Equation

Given $B_\nu(\tau)$, this equation can be solved for J_ν, and then the source function is determined from B_ν and J_ν. To solve the radiative diffusion equation, we need boundary conditions. These can be specified in the two-stream approximation, as shown on the next page

Approximate the specific intensity with two rays, I_ν^+ and I_ν^-, traveling in the directions $\mu = {}^{+1}/\sqrt{3}$ and ${}^{-1}/\sqrt{3}$, respectively.



Now write the moments of the specific intensity using these two discrete rays.

$$J_\nu = \frac{1}{2}\left(I_\nu^+ + I_\nu^-\right) \qquad (1)$$

$$F_\nu = 2\pi\left(\frac{1}{\sqrt{3}} I_\nu^+ - \frac{1}{\sqrt{3}} I_\nu^-\right) \qquad (2)$$

$$P_\nu = \frac{2\pi}{c}\left(\frac{1}{3} I_\nu^+ + \frac{1}{3} I_\nu^-\right) = \frac{4\pi J_\nu}{3c} = \frac{u_\nu}{3}$$

The last equation above is the Eddington approximation so the two-stream approximation with the choice $\mu = \pm 1/\sqrt{3}$ equivalent to the Eddington approximation. Now solve Equations (1) and (2) for I_ν^+ and I_ν^-

$$I_\nu^+ = J_\nu + \frac{\sqrt{3}}{4\pi} F_\nu$$
$$I_\nu^- = J_\nu - \frac{\sqrt{3}}{4\pi} F_\nu$$

But the first moment equation on the previous page, plus the Eddington approximation, give $F_\nu = \frac{4\pi}{3}\frac{dJ_\nu}{d\tau_\nu}$

Therefore

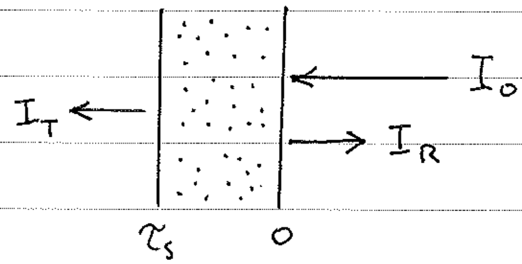
$$I_\nu^+\left(\tau_\nu, \mu = \tfrac{1}{\sqrt{3}}\right) = J_\nu + \frac{1}{\sqrt{3}}\frac{dJ_\nu}{d\tau_\nu}$$

$$I_\nu^-\left(\tau_\nu, \mu = \tfrac{-1}{\sqrt{3}}\right) = J_\nu - \frac{1}{\sqrt{3}}\frac{dJ_\nu}{d\tau_\nu}$$

These can be used to specify two boundary conditions on J_ν.

Example:

Two-Stream Approximation for Pure Scattering Layer



$$\boxed{I_0 = I_T + I_R}$$

Boundary conditions are:

$$J(0) = \frac{1}{2}(I_0 + I_R)$$

$$J(\tau_s) = \frac{1}{2} I_T$$

Two-stream equations are:

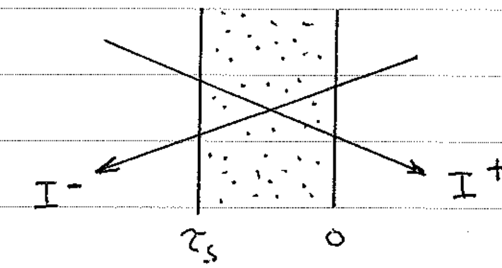
$$I_R = J(0) + \frac{1}{\sqrt{3}} \frac{dJ}{d\tau} \qquad (1)$$

$$I_T = J(\tau_s) - \frac{1}{\sqrt{3}} \frac{dJ}{d\tau} \qquad (2)$$

From equation (2) and B.C.

$$I_T = \frac{1}{2} I_T - \frac{1}{\sqrt{3}} \frac{dJ}{d\tau}$$

$$\boxed{I_T = -\frac{2}{\sqrt{3}} \frac{dJ}{d\tau}}$$



$$\boxed{\begin{aligned} I^+(\tau) &= J(\tau) + \frac{1}{\sqrt{3}} \frac{dJ}{d\tau} \\ I^-(\tau) &= J(\tau) - \frac{1}{\sqrt{3}} \frac{dJ}{d\tau} \end{aligned}}$$

In the Eddington approx.

$$\frac{1}{3} \frac{d^2 J}{d\tau^2} = J - S$$

where $S_\nu = \dfrac{\sigma_\nu J + \alpha_\nu B_\nu}{\sigma_\nu + \alpha_\nu}$

For pure scattering $S_\nu = J_\nu$

and $\dfrac{d^2 J}{d\tau^2} = 0$

$\Rightarrow$ J is a linear function of τ_s

$$\boxed{J(\tau_s) = J(0) + \tau_s \frac{dJ}{d\tau_s}}$$

Subtract eq. (2) from eq. (1)

$$I_R - I_T = J(0) - J(\tau_s) + \frac{2}{\sqrt{3}} \frac{dJ}{d\tau}$$

$$I_R - I_T = -\tau_s \frac{dJ}{d\tau} + \frac{2}{\sqrt{3}} \frac{dJ}{d\tau}$$

$$I_R - I_T = -\tau_s \frac{dJ}{d\tau} - I_T$$

$$\boxed{I_R = -\tau_s \frac{dJ}{d\tau}}$$

$$\Rightarrow \boxed{\frac{I_T}{I_R} = \frac{2}{\sqrt{3}\,\tau_s} \quad \text{or} \quad I_T = \frac{I_0}{1 + \frac{\sqrt{3}\,\tau_s}{2}}}$$