Inverse Compton Scattering (of photons by relativistic electrons)

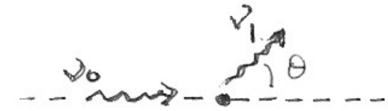
We will show that inverse compton scattering boosts a photon's energy by a factor $\sim \gamma^2$, and that the expressions for total power radiated in synchrotron and inverse compton radiation are almost the same.

Prelude: Remember the Thomson cross-section (differential) (unpolarized)

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 (1+\cos^2\theta) \qquad \text{where } r_0 = \frac{e^2}{mc^2}$$

The Thomson cross section is only valid in the limit $h\nu \ll mc^2$, namely $h\nu \ll 511$ keV. In general, the Klein-Nishina cross section should be used, but it is dependent on the energy of the scattered and incident photons, and it is not symmetric forward and back:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 \left(\frac{\nu_1}{\nu_0}\right)^2 \left(\frac{\nu_0}{\nu_1} + \frac{\nu_1}{\nu_0} - \sin^2\theta\right)$$



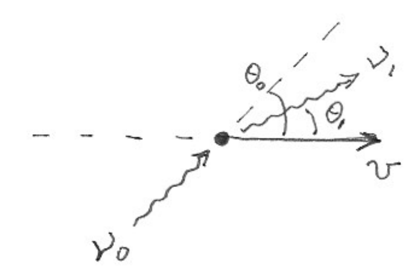
$$\nu_1 = \frac{\nu_0}{1+\frac{h\nu_0}{mc^2}(1-\cos\theta)}$$

or $\lambda_1 - \lambda_0 = \frac{h}{mc}(1-\cos\theta)$

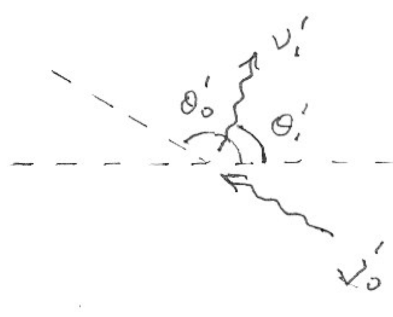
Note that $d\sigma/d\Omega$ reduces to the Thomson limit when $\nu_1 \approx \nu_0$. This limit corresponds to $\lambda_0 \gg h/mc$, where h/mc is called the Compton wavelength $\lambda_c = h/mc = 0.02426$ Å

Single scattering: Now consider scattering from a moving electron. We will assume the non-relativistic (Thomson) cross-section applies in the rest frame of the electron, so $h\nu' \ll mc^2$. This means that $\nu_1' \approx \nu_0'$

lab frame (S)



electron rest frame (S')



Note that θ is the angle between $\vec{v}$ and the direction of propagation of the photon. Use the Doppler formula to relate photon frequencies in electron frame and "lab" frame.

$$\nu_0' = \nu_0 \gamma (1 - \beta \cos\theta_0)$$

In S' $\nu_1' = \nu_0'$ by assumption

So $\nu_1' = \nu_0' = \nu_0 \gamma (1 - \beta \cos\theta_0)$ (1)

Now apply the inverse Doppler transformation to get ν_1 in the lab frame

$$\nu_1 = \nu_1' \gamma (1 + \beta \cos\theta_1')$$

So $\nu_1 = \nu_0 \gamma^2 (1 - \beta \cos\theta_0)(1 + \beta \cos\theta_1')$

We will show that a proper average over an isotropic distribution of photons in the lab frame gives

$$\langle \nu_1 \rangle \approx \frac{4}{3} \gamma^2 \nu_0$$

Evaluate the power in photons scattered by <u>a single electron in its rest frame</u>

$P' = \sigma_T c \int E_0' \, n'(E_0') \, dE_0'$ (2)

where $E_0' = h\nu_0'$

Distribution function:

and $n(E_0')$ is the photon energy distribution $\left[\frac{\text{photons}}{\text{energy} \cdot \text{vol}}\right]$

(This integral is the generalization of the power scattered from monochromatic photons $P = \underbrace{n \sigma_T c}_{\text{rate of scattered photons } [s^{-1}]} E$)

Now we need to use the fact that

$\frac{n(E)\, dE}{E}$ is a Lorentz invariant. To prove this...

Note that $n(E)\,dE = f(\vec{p})\,d^3\vec{p}$

where $f(\vec{p})$ is the phase space distribution function, which has the dimensions of photons/vol/momentum. Previously, we showed that f is a Lorentz invariant, and also that

$$d^3p = \gamma\, d^3p'$$

But we also know that $E = \gamma E'$ so d^3p transforms the same way as E. This means that

$$\frac{n(E)\,dE}{E} = \frac{f(p)\,d^3p}{E} \quad \text{is a Lorentz invariant}$$

$$\frac{n'(E')\,dE'}{E'} = \frac{n(E)\,dE}{E}$$

This is useful because we can write equation (2) for the power as

$$P' = \sigma_T c \int E_0'\, n'(E_0')\,dE_0' = \sigma_T c \int (E_0')^2 \frac{n'(E_0')\,dE_0'}{E_0'}$$

$$= \sigma_T c \int (E_0')^2 \frac{n(E_0)\,dE_0}{E_0}$$

Now use the Doppler shift (equation 1) to replace E_0'

$$P' = P = \sigma_T c\, \gamma^2 \int E_0^2 (1-\beta\cos\theta)^2 \frac{n(E_0)\,dE_0}{E_0}$$

$$= \sigma_T c\, \gamma^2 \int E_0 (1-\beta\cos\theta)^2\, n(E_0)\,dE_0$$

Isotropic Distribution

For an isotropic distribution of photons

$$\langle(1-\beta\cos\theta)^2\rangle = \frac{1}{2}\int_0^\pi (1-\beta\cos\theta)^2 \sin\theta\, d\theta = \frac{1}{2}\int_{-1}^{1} (1-\beta x)^2\,dx$$

$$= 1 + \frac{1}{3}\beta^2$$

So $P = \sigma_T c\, \gamma^2 (1 + \frac{1}{3}\beta^2)\, U_{ph}$ where $U_{ph} = \int E_0\, n(E_0)\,dE_0$

the photon energy density

This is the power "emitted" by the electron, while the power absorbed is just $\sigma_T c U_{ph}$ - Therefore the net Inverse Compton power is:

$$P_{IC} = \sigma_T c U_{ph} [\gamma^2(1 + \tfrac{1}{3}\beta^2) - 1]$$

$$= \sigma_T c U_{ph} (\gamma^2 - 1 + \tfrac{1}{3}\gamma^2\beta^2)$$

$$= \sigma_T c U_{ph} (\gamma^2\beta^2 + \tfrac{1}{3}\gamma^2\beta^2)$$

$$\frac{1}{1-\beta^2} - 1 = \gamma^2\beta^2$$

$$\boxed{P_{IC} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{ph}}$$

Note that this is identical to the previously derived synchrotron power

$$P_{synch} = \frac{4}{3}\sigma_T c \gamma^2\beta^2 U_{mag} \qquad \text{where } U_{mag} = B^2/8\pi$$

where we have only to replace U_{mag} with U_{ph} (photon energy density).

This also means that the lifetime of an electron to loss of energy by either process can be simply related to the energy density in either the photons or the B field

$$t = \frac{\gamma m c^2}{P} \qquad \boxed{t_{IC} = \frac{3\times10^7}{\gamma U_{ph}}} \qquad \boxed{t_{synch} = \frac{3\times10^7}{\gamma U_{mag}}} \qquad [s]$$

All of these relations apply to emission by a single electron. Just as we did for synchrotron radiation, we can calculate the spectrum of inverse Compton radiation. Remember that we are still assuming that $h\nu' \ll mc^2$ in the electron rest frame so that we can ignore the small energy shift given by $\lambda_1 - \lambda_0 = (h/mc)(1 - \cos\theta)$. Also, we approximate the scattering in the rest frame as isotropic, which is valid if the electrons and photons in the "lab" are isotropic

$$\frac{d\sigma'}{d\Omega'} = \frac{1}{4\pi}\sigma_T = \frac{2}{3}r_0^2 \qquad \left(\text{instead of } \tfrac{1}{2}r_0^2(1+\cos^2\theta)\right)$$

What is the spectrum from the scattering of photons of a single energy E_0 from electrons of a single γ?

Inverse Compton Scattering Spectra - Summary only

We use the Thomson limit in the rest frame of the electron to calculate the spectrum of scattered radiation from an isotropic distribution of electrons of velocity β and density n exposed to an isotropic flux of photons F_0 with energy E_0

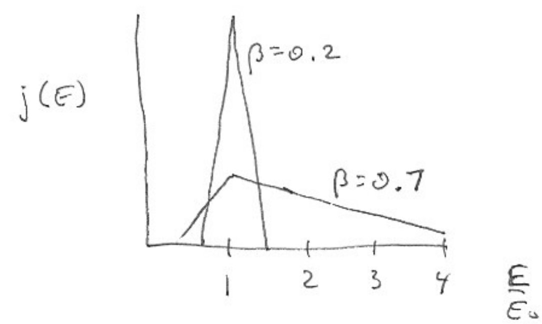
$F_0 \left[\frac{\text{photons}}{\text{cm}^2\ \text{s ster}}\right] \qquad n \left[\frac{\text{electrons}}{\text{cm}^3}\right]$

$$j(E) = \begin{cases} \dfrac{n\sigma_T F_0}{4\gamma^2\beta^2 E_0}\left[(1+\beta)\dfrac{E}{E_0} - (1-\beta)\right] & \text{if } \dfrac{1-\beta}{1+\beta} < \dfrac{E}{E_0} < 1 \\[2ex] \dfrac{n\sigma_T F_0}{4\gamma^2\beta^2 E_0}\left[1+\beta - (1-\beta)\dfrac{E}{E_0}\right] & \text{if } 1 < \dfrac{E}{E_0} < \dfrac{1+\beta}{1-\beta} \end{cases}$$

The emission coefficient $j(E)$ has units $\left[\frac{\text{photons}}{\text{cm}^3\ \text{s ster erg}}\right]$

That is $j(E) = j_\nu / h^2\nu$

As β approaches 1 almost all of the emission is boosted in energy, and the lower of the two expressions for $j(E)$ dominates.



remember $\beta^2 = 1 - \frac{1}{\gamma^2}$

$\beta \approx 1 - \frac{1}{2\gamma^2}$ as $\beta \to 1$

Then

$$\boxed{j(E) = \frac{1}{2}\frac{n\sigma_T F_0}{\gamma^2 E_0}\left(1 - \frac{1}{4\gamma^2}\frac{E}{E_0}\right)}$$

Inverse Compton spectrum from a power-law distribution of Electron energies

Let $n(\gamma)\,d\gamma = n_0 \gamma^{-p}\, d\gamma$

The specific emissivity is $4\pi E\, j(E)$ $\left[\frac{erg}{cm^3\, s\, erg}\right]$

$$4\pi E\, j(E) = 4\pi \frac{1}{2} n_0 \sigma_T F_0 \frac{E}{E_0} \int_{\gamma_1}^{\gamma_2} \frac{\gamma^{-p}}{\gamma^2} \left(1 - \frac{1}{4\gamma^2} \frac{E}{E_0}\right) d\gamma$$

Now replace the flux F_0 of photons of single energy E_0 with a spectrum of incident photons $I(E_0)$, and integrate over E_0

$$4\pi E\, j(E) = 4\pi \frac{n_0 \sigma_T}{2} \int \frac{E}{E_0} I(E_0)\, dE_0 \int_{\gamma_1}^{\gamma_2} \gamma^{-p-2} \left(1 - \frac{1}{4\gamma^2} \frac{E}{E_0}\right) d\gamma$$

Let $x = \frac{1}{4\gamma^2} \frac{E}{E_0}$ and change variable from γ to x

$$\gamma^{-p-2} d\gamma = -\left(\frac{4 x E_0}{E}\right)^{p/2+1} \frac{1}{4} \left(\frac{E}{E_0}\right)^{1/2} x^{-3/2}$$

$$4\pi E\, j(E) = 2^{p+1} \pi n_0 \sigma_T E^{-(p-1)/2} \int E_0^{(p-1)/2} dE_0\, I(E_0) \int_{x_1}^{x_2} x^{(p-1)/2} f(x)\, dx$$

where $x_1 = \frac{E}{4\gamma_1^2 E_0}$, $x_2 = \frac{E}{4\gamma_2^2 E_0}$, $f(x) = (1 - x)$

Suppose that $\gamma_2 \gg \gamma_1$. Then the integral over x is only a function of p.

The spectral index of the radiation is $s = \frac{p-1}{2}$

The spectrum of inverse Compton radiation from a power-law distribution of electrons of index p, is a power law of index $s = \frac{p-1}{2}$