Numerically solving ODE’s

Lecture 5
Differential Equations

- Ordinary Differential Equations (ODE)
  - Function(s) of one independent variable (e.g. $f(x)$, $g(t)$)

- Partial Differential Equations
  - Function(s) of two or more independent variables: $f(x,y)$

- Can have coupled sets of ODE
  - E.g. $f(x)$, $g(x)$ depend on each other

- ODE’s (relatively) easy to solve
  - PDE’s can be harder
The Pendulum

F = ma in direction of $\theta$:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

If $\theta$ is small, then $\sin\theta \sim \theta$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Initial conditions $\theta = \theta_0$ at $t=0$

Solution: $\theta = \theta_0 \cos((g/l)^{1/2} \ t)$
The linear Pendulum

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{l} \theta = 0 \]

Initial conditions \( \theta = \theta_0 \) at \( t=0 \)

Solution: \( \theta = \theta_0 \cos((g/l)^{1/2} t) \)
The (non-linear) Pendulum

For large $\theta$, the ODE is non-linear:

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

How do we solve this?
Euler’s method

Take a general first-order ODE:

$$\frac{dx}{dt} = f(x,t)$$

Approximate \(x(t_1=h)\) as:

$$x_1 = x_0 + hf(x_0,0)$$

Approximate \(x(t_2=2h)\) as:

$$x_2 = x_1 + hf(x_1,h)$$

and so on…

In general:

$$x_{n+1} = x_n + hf(x_n,t_n)$$
Apply Euler to Pendulum

2\textsuperscript{nd} order ODE:

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{l} \sin \theta = 0 \]

First, re-write in dimensionless form:

Define new time variable:

\[ \tau = t(l/g)^{-1/2} \]

To get:

\[ \frac{d^2 \theta}{d\tau^2} + \sin \theta = 0 \]
Apply Euler to Pendulum

2nd order ODE:

\[ \frac{d^2 \theta}{d\tau^2} + \sin \theta = 0 \]

Split into two coupled 1st order ODE’s:

\[ \frac{d\theta}{d\tau} = y \]

\[ \frac{dy}{d\tau} = -\sin(\theta) \]

Apply Euler’s method to both equations
from math import *

def euler(dt, y, theta):
    dtheta = dt * y
    dy = dt * (-sin(theta))
    y = y + dy
    theta = theta + dtheta
    return (y, theta)

theta = 0.5
y = 0
t = 0
tstop = 40.0
dt = 0.05

while (t < tstop):
    (y, theta) = euler(dt, y, theta)
    t = t + dt
    print t, theta, y
Apply Euler to Pendulum

800 steps
Apply Euler to Pendulum

800 steps

4000 steps
Phase space diagram

![Phase space diagram](image)

Pendulum at one moment in time
Phase space diagram

800 steps
Improved Euler’s method

Instead of using slope just at $x_n$:

$$x_{n+1} = x_n + hf(x_n, t_n)$$

Use an average slope:

$$x_{n+1} = x_n + \frac{1}{2} h(f(x_n, t_n) + f(x_{n+1}, t_{n+1}))$$

But $x_{n+1}$ appears on RHS?!

Solution: use Euler method to get an approximate $x_{n+1}$
**Improved Euler’s method**

<table>
<thead>
<tr>
<th>Original Euler method</th>
<th>Improved Euler method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c = hf(x,t)$</td>
<td>$c = hf(x,t)$</td>
</tr>
<tr>
<td>$x_{\text{new}} = x + c$</td>
<td>$d = hf(x+c, t+h)$</td>
</tr>
<tr>
<td></td>
<td>$x_{\text{new}} = x + (c+d)/2$</td>
</tr>
</tbody>
</table>
Apply Improved Euler to Pendulum
Improved Euler:
Phase space diagram

800 steps
Euler

800 steps
Improved Euler

4000 steps
Euler (red)
High amplitude pendulum
High amplitude pendulum
High amplitude pendulum
High amplitude pendulum
High amplitude pendulum
Summary

• Euler method
  – slope at $x_n$  
  Error $\sim h$

• Improved Euler
  – slope averaged over 2 points
  – Error $\sim h^2$

• Runge-Kutta
  – Slope averaged over 4 points
  – Error $\sim h^4$

• Phase diagram (velocity vs. position)