5.3 Growth of Perturbations

In the last section, we saw the utility of looking at the growth of fluctuations in terms of the amplitudes of plane waves with a given wavelength (i.e. Fourier modes). In this way, we were able to convert a spatial gradient into a simple multiplication by $k^2$ and so write down an ordinary differential equation for how the amplitude of the wave evolves. This means that — for small perturbations — the amplitude of each mode evolves independently from the others. In fact, quite generally, it is very useful to parameterize the perturbation field $\delta$ in terms of Fourier modes.

We can link the wavenumber $k$ of a given perturbation to an approximate length scale via $r = 2\pi/k$ and, even more approximately, to a mass: $M \sim \bar{\rho} r^3 \sim \bar{\rho} k^{-3}$. If we recall the expression we obtained earlier for the correlation function:

$$\xi(r) = \frac{V}{2\pi^2} \int P(k) \frac{\sin kr}{kr} k^2 dk$$

(55)

We can integrate this exactly for a given power spectrum, but to get a better feeling of the link between the power spectrum and the correlation function (which measures the amplitude of fluctuations on a given length scale), we can do this approximately by noting that the $\sin kr/kr$ term is a bit like a window function that is approximately constant at small $k$, and then goes to zero beyond $k \sim 1/r$. Taking $P(k) \sim k^n$, this gives:

$$\xi(r) \sim r^{-(n+3)}$$

(56)

or, in terms of mass: $\xi(M) \sim M^{-(n+3)/3}$.

So we have seen that the power spectrum $P(k)$ determines the clustering properties and that the evolution of the power spectrum in the linear regime breaks down to the problem of evolving the amplitude of each mode independently. Of course, we still haven’t answered what the initial amplitude of each mode should be — an ordinary differential equation needs an initial condition in order to specify the final state. The usual choice of an initial power spectrum is the “Harrison-Zeldovich" spectrum: $P(k) = A k^n$, where $n = 1$ and $A$ is a constant. This is a natural choice because it means that the size of given perturbation is always the same when it enters the Universe’s horizon (basically the light-travel time given the Universe’s age).

This choice has also been verified by observations of the microwave background by the WMAP satellite ($n$ is 1 to within a few percent). However, it doesn’t mean that the current power spectrum looks like this because each mode doesn’t necessarily grow after entering the horizon. During the period when the universe was dominated by the energy density of the CMB, the perturbations in the photons dominated. Since the speed of sound for a photon fluid is $1/3$ the speed of light, the Jeans length for the photon gas is very large, nearly the horizon length. It was not until the energy in the photon fluid redshifted away at $z \sim 10^4$ that the dark-matter density dominated and perturbations smaller than the horizon-size could grow again. This resulted in a processing of the spectrum, such that at small length scales the spectrum of fluctuations in nearly flat. In terms of the correlation function $\xi(M) M^0$ or $n \sim -3$. On the opposite end, the largest perturbations entered the horizon after matter domination and retained their $P(k) \sim k$ shape. The result is a spectrum that bends slowly from $k^1$ at small $k$ (large $r$) to nearly $k^{-3}$ at large $k$ (small $r$). The exact shape of the spectrum can be computed by solving equations like the one we derived earlier and is generally written in the form

$$P(k) = AD^+(t) k^n T_f^2(k)$$

(57)

where $T_f(k)$ is known as the transfer function and its shape depends on the cosmological parameters assumed. We have assumed that each mode is larger than its Jeans length so that they all grow according to a single function $D^+(t)$ which depends only on the values of the cosmological parameters.

As a given mode grows, it eventually leaves the linear regime and when $\xi(r) \sim 1$ it is considered fully non-linear. Some time before this, the linear treatment we have given breaks down (although note that modes with smaller amplitudes can still be treated by the perturbation theory). Because of the shape of the spectrum, modes with short wavelengths “break” first. This is the origin of the statement that CDM-like models are hierarchical: small structures collapse first and then combine to form larger objects. The scale where $\xi \sim 1$ is often called the non-linear scale. With this, we can re-write the correlation function as

$$\xi(M) = (M/M_{NL})^{-(n+3)/3}$$

The next step is to go beyond the linear regime and develop simple models that can predict when a perturbation collapses. As we will see, it is useful to link the two so that we can go all the way from initial perturbations to a distribution of collapsed object.
5.4 Spherical collapse model and linear theory

Earlier, we described one of the solutions for a closed universe in terms of parametric solution featuring cosines (equation 43). By choosing the center properly, we can re-interpret this in terms of the behaviour of a spherical perturbation of uniform density with radius $R$ (once again the justification for using Newtonian gravity comes from Birkhoff’s theorem):

$$\frac{R}{R_m} = \frac{1}{2} (1 - \cos \eta)$$
$$\frac{t}{t_m} = \frac{1}{\pi} (\eta - \sin \eta).$$

(58)

In order to relate this to linear theory, we need to know what the linear prediction would be for such a spherical perturbation. First we find the relation between $R$ and $t$ for small values of $\eta$ (i.e. when linear theory should still hold). This can be done by expanding $R(\eta)$ and $t(\eta)$ with a Taylor series expansion:

$$\frac{R}{R_m} = \frac{1}{2} \left[ 1 - \left( 1 - \frac{\eta^2}{2} + \frac{\eta^4}{24} + \ldots \right) \right]$$
$$\frac{t}{t_m} = \frac{1}{\pi} \left[ \eta - \left( \eta - \frac{\eta^3}{6} + \frac{\eta^5}{120} + \ldots \right) \right].$$

(59)

We take the second equation above and solve for $\eta$ in terms of $t$ to get:

$$\eta^2 = \left( \frac{6\pi t}{t_m} \right)^{2/3} \left( 1 + \frac{1}{30} \left( \frac{6\pi t}{t_m} \right)^{2/3} + \ldots \right).$$

(60)

This can then be substituted into the first equation to get $R$ as a function of $t$:

$$\frac{R}{R_m} = \frac{1}{4} \left( \frac{6\pi t}{t_m} \right)^{2/3} \left( 1 + \frac{1}{30} \left( \frac{6\pi t}{t_m} \right)^{2/3} - \frac{1}{12} \left( \frac{6\pi t}{t_m} \right)^{2/3} + \ldots \right)$$
$$\approx \frac{1}{4} \left( \frac{6\pi t}{t_m} \right)^{2/3} \left( 1 + \frac{1}{20} \left( \frac{6\pi t}{t_m} \right)^{2/3} \right) \ldots$$

(61)

The term outside the parenthesis is the unperturbed evolution (recall that for a flat universe, $R(t) \sim t^{2/3}$). To relate this to something we have worked out before, that is the evolution of the overdensity $\delta$ as a function of time for linear theory, we can write:

$$\delta_{\text{lin}}(t) = \frac{\rho}{\bar{\rho}} - 1 = \frac{(R/R_m)^3}{(R/R_m)^3_{\text{linear}}} - 1 = \left( 1 - \frac{1}{20} \left( \frac{6\pi t}{t_m} \right)^{2/3} \right)^{-3} - 1$$
$$\approx \frac{3}{20} \left( \frac{6\pi t}{t_m} \right)^{2/3}.$$

(62)

This means that when the spherical model has reached full collapse at $t = 2t_m$, the linear calculation indicates that the overdensity should be $\delta_{\text{linear}} = 3/20(12\pi)^{2/3} \approx 1.686$. Besides confirming our expectation that the $\delta \sim 1$ indicates the full non-linear phase, this expression lets us tie together the full non-linear and linear calculations.

The spherical collapse model predicts that the density is formally infinite at the point of collapse. This is unrelatistic because non-radial motions would have grown to the point that the cannot be ignored. One simple approximation is to assumed that the collapse will stop when the virial theorem is fulfilled, which translates into an overdensity of a few hundred.

We should also point out that there are a number of other non-linear approximations, including variations of the spherical model such as ellipsoidal collapse. Even in terms of perturbation theory, it is possible to do somewhat better by using Lagrangean perturbation theory, that is to describe the perturbation by the change of the comoving position of a fluid element rather than the change in the density at fixed position.