Part II Galaxy Formation Theory

It is common to divide ideas about galaxy formation into two categories: “monolithic collapse” in which a galaxy formed at early times from a single cloud, and “hierarchical formation” in which a galaxy is built out of merging many smaller units together. While it is often useful to make this distinction, it is surely an over-simplification. In these notes, we will focus mostly on galaxy formation within current theories of cosmological evolution, since this framework has proved extremely successful in describing a wide range of observations.

5 Cosmology

This is not intended to be a comprehensive review of cosmological theory, but instead we will concentrate on developing the tools required for understanding galaxy formation.

5.1 Homogeneous cosmology

The first step is to present a mini-review of our theory describing the evolution of the universe as a whole. This is dealt with in many texts so we present just the bare-bones here. The primary tool is the Friedman equation which describes the evolution of the scale factor $a(t)$ as a function of time. General Relativity is required to describe an infinite homogeneous universe, but a quasi-Newtonian approach can be useful. In this, we write down the energy conservation equation for a spherical shell of material with density $\rho$ at some distance $r$ from its center (which could be any point):

$$\frac{1}{2}(\rho 4\pi r^2dr)\dot{r}^2 - \frac{GM_r}{r}(\rho 4\pi r^2dr) = C dr$$

(40)

Here, $C$ is a constant and $M_r$ indicates the mass within radius $r$, which is also a constant for a homogeneous universe. If we define comoving coordinates such that $x = r/a(t)$ where $x$ is fixed for any given shell, then $\dot{a} = x\dot{r}$ (in the homogeneous case), and we can re-write this equation in the more familiar form for the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\bar{\rho}}{3} + \frac{\kappa}{a^2} = \frac{\Lambda}{3}.$$  

(41)

We have redefined the total energy in terms of a new constant $\kappa$ and added the cosmological constant $\Lambda$ on the right-hand side. The cosmological constant is outside of the quasi-Newtonian derivation and represent a gravitational repulsion that grows with distance. This term is allowed but often not included in the Einstein field equations. Einstein introduced to allow for a static solution ($\dot{a} = 0$) but quickly disavowed it both because the static solution is not stable and observational evidence indicates the universe is expanding. Recent evidence indicates that some such term is currently causing accelerated expansion in the current universe.

The Friedmann equation can be re-written in terms of dimensionless parameters as:

$$H^2(t) = H_0^2 \left[ \Omega_\Lambda + \Omega_m a^{-3} - \Omega_k a^{-2} \right]$$

(42)

where $H = \dot{a}/a$ is the Hubble constant and $H_0$ is its current value. $\Omega_m$ is the (non-relativistic) mass density of the universe in terms of the density required to close the universe (i.e. for $\kappa = 0$); similarly $\Omega_{\Lambda lambda}$ represents the cosmological constant contribution and $\Omega_k$ is the curvature term. Note that $\Omega_\Lambda + \Omega_m + \Omega_{\lambda k} = 1$.

Obviously, for given values of the parameters, we can solve this equation for $a(t)$. In general, closed-form representation do not exist, but there are some cases which are easily to solve.

The first is the case of a flat universe in which $\kappa = 0$ and $\Lambda = 0$. In thise the solution is trivially $a(t) = (t/t_0)^{2/3}$ where $t_0$ is the current age of the universe and we have taken the usual notation $a(t_{now}) = 1$ (for this choice it can be shown that the redshift is just related to $a$ via $a = 1/(1+z)$).
The next case still keeps \( \Lambda = 0 \), but now we take \( \kappa < 0 \) for which the universe is closed. In this case we can write down the parametric solution:

\[
\frac{a}{a_m} = \frac{1}{2} (1 - \cos \eta) \\
\frac{t}{t_m} = \frac{1}{\pi} (\eta - \sin \eta)
\] (43)

The so-called development angle \( \eta \) has no independent physical meaning but goes from 0 when \( a(t_m) = a_m \) reaches its maximum extent at which point \( \dot{a} = 0 \) and the universe starts to recollapse. Re-collapse to a single point when \( \eta = 2\pi \). When the universe is open \( \kappa > 0 \) the solution above still holds with cos replaced by cosh (and sin by sinh).

### 5.2 Inhomogeneous cosmology

Of course, the universe is not completely smooth. It appears to be smooth on large scales and was smooth at early times, but inhomogeneities are the spice of life. To study them, we start with the usual fluid equations plus gravity:

\[
\frac{\partial \rho}{\partial t}_r + \nabla_r \cdot (\rho \mathbf{u}) = 0 \\
\frac{\partial \rho}{\partial t}_r + (\mathbf{u} \cdot \nabla_r) \mathbf{u} = -\nabla_r \phi - \frac{1}{\rho} \nabla_r p
\] (44)

\[
\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta.
\]

The first equation represents the conservation of momentum and the second the conservation of momentum. These can be derived by taking moments of the collisionless Boltzmann equation. The gravitational potential \( \phi \) comes from Poisson’s equation and the pressure \( p \) we leave unspecified here; it comes in general from the equation of state and depends on the gas properties.

We have included a subscript \( r \) on the derivatives to emphasize the fact that these are all to be taken at constant physical co-ordinate \( r \). However, it makes more sense in an expanding universe to transform into a set of coordinates which are comoving with the expansion \( \mathbf{x} = \mathbf{r}/a \). Unlike the homogeneous case, the comoving coordinate is not constant so the physical velocity \( \mathbf{u} = \dot{\mathbf{r}} \) is related to the comoving velocity via:

\[
\mathbf{u} = \dot{\mathbf{r}} = a \dot{x} + \dot{a} \mathbf{x}
\] (45)

The first term on the right-hand side is often known as the peculiar velocity \( \mathbf{v} = a \dot{x} \) and the second term the Hubble velocity. The co-ordinate transformation has two parts. First, the spatial derivative in comoving coordinates \( \nabla \) is related to the physical operator by \( \nabla = a \nabla_r \). The second is the time derivative, which becomes:

\[
\frac{\partial \rho}{\partial t}_x = \frac{\partial \rho}{\partial t}_r + \frac{\partial \mathbf{r}}{\partial t} \cdot \nabla_r \rho = \frac{\partial \rho}{\partial t}_r + \dot{a} \frac{\dot{x}}{a} \nabla_r \rho.
\] (46)

On the right-hand side, the first term is the rate of change of \( \rho \) at constant \( r \) and the second is the change in \( \rho \) due to the fact that for constant \( x \), the value of \( r \) is changing. Putting in these definition results in the comoving co-ordinate fluid equations.

One important application of these equations is an equation for the evolution of small perturbations around a smooth universe. To do this, we re-write the density as \( \rho(x) = \bar{\rho}(1 + \delta(x)) \) where eventually (but not yet), we will take \( \delta \) to be small compared to unity. Adopting this definitions, the fluid equations plus Poisson equation become:

\[
\frac{\partial \delta}{\partial t}_x + \frac{1}{a} \nabla \cdot [(1 + \delta) \mathbf{v}] = 0 \\
\frac{\partial \mathbf{v}}{\partial t}_x + \dot{a} \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \phi - \frac{1}{a \rho} \nabla p
\] (47)

\[
\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta.
\]
We have redefine the potential $\phi$ to remove the terms arising solely from the expansion. Notice the appearance of a new term on the right-hand side of the momentum equation. This drag-like term (it acts as a force opposing the direction of motion) arising solely from the coordinate system (and so it is a fictitious force) and acts to damp peculiar velocities in the absence of other forces.

Now taking the perturbation to be small ($\delta << 1$), we can drop all terms which are not linear in $\Delta$ and $v$ (which should also be small compared to $v_H$):

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} \mathbf{v} = -\nabla \phi - \frac{1}{a \rho} \nabla p$$

(48)

(49)

and the potential equation is unchanged. Notice that we have dropped an entire non-linear term in the momentum equation. This allows us to take the divergence of the second equation and solve the first for $\nabla \cdot \mathbf{v}$, eliminating it from the second equation. The resulting $\nabla^2 \phi$ term on the right-hand side and be replaced by the poisson equation to obtain a relation that only involved $\delta$. We replace the pressure term with an equation of state that depends only on density $p = c_s^2 \rho$, where $c_s$ is the speed of sound for this equation of state (recall that $c_s^2 = dP/d\rho$). The result is (dropping the subscript $x$ notation):

$$\frac{\partial^2 \delta}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial \delta}{\partial t} = 4 \pi G \bar{\rho} \delta + \frac{c_s^2}{a^2} \nabla^2 \delta.$$  

(50)

This is the evolution equation for small perturbations. If we ignore for the moment the pressure term, then there are no spatial derivatives so $\delta(x)$ evolves in a purely local fashion and the partial derivatives become full derivatives. The resulting ordinary differential equation is like the one that describes simple harmonic motion except the restoring force is in the opposite direction so the system exhibits runaway (gravitational) growth. For an Einstein-deSitter universe (with $\Lambda = 0$ and $\kappa = 0$), the solution is simply:

$$\delta = At^{2/3} + Bt^{-1}$$  

(51)

where $A$ and $B$ are constants. The first term is called the growing mode and at late times it must dominate so we focus on the growing mode solution. Notice that it has the same time-dependence as the scale factor ($\delta(t) \propto a(t)$). Note also that the growth rate differs from the exponential that one would generally find for a static universe. Thus, an expanding universe slows gravitational instability.

In the general universe case, for any values of the cosmological parameters, we write:

$$\delta(x, t) = D^+(t) \delta_0(x)$$

(52)

where $D^+$ is the growing mode solution for the given cosmological parameters and depends only on $x$.

If we cannot neglect the pressure, then we can still derive an ordinary differential equation but first we have to expand $\delta(x)$ in terms of a set of plane-waves with wave-numbers $k$ (i.e. perform a Fourier transform). If we do this, then the single evolution equation turns into a series of ordinary differential equations for each wave-number $k$:

$$\ddot{\delta}_k + 2 \frac{\dot{a}}{a} \dot{\delta}_k = 4 \pi G \bar{\rho} \delta_k - \frac{k^2 c_s^2}{a^2} \delta_k$$

(53)

where we use the notation $\delta_k$ for the amplitude of a plane-wave with a given wave-number. The left-hand side is again like simple-harmonic motion with the right-hand side being the restoring force. If the pressure-term is larger so that the right-hand side is negative, then the wave oscillates. Otherwise, it grows as before. When the right-hand side is exactly zero then gravitational forces balance pressure forces and this occurs at the (comoving) Jean’s length:

$$\lambda_J = \frac{2\pi}{k_J} = \left(\frac{\pi c_s^2}{G \bar{\rho}}\right)^{1/2}$$

(54)

We have derived these equations for a single fluid, but the universe is really made up of (at least) several components: baryons, photons, neutrinos and dark matter. It is straightforward to develop equations describing the evolution of various components.