Note that we have dropped the $\partial \phi / \partial R$ and $\partial \phi / \partial z$ terms because the unperturbed orbit is at a minimum in the potential (and the constant term can be set to zero because it has no physical meaning anyway). Putting this into the equation for $\ddot{R}$ (similar results apply for $\ddot{z}$), and noting that $\ddot{R} = \ddot{x}$, we find:

$$\ddot{x} = -\kappa^2 x$$

which describes simple harmonic motion with a frequency known as the epicyclic frequency:

$$\kappa^2 = \frac{\partial^2 \phi}{\partial R^2} + \frac{3L_z^2}{R_g^4}$$

The epicyclic approximation is useful because it describes the timescale for typical non-circular perturbations to oscillate (or grow) in the radial direction of a disk.

If we describe the rotation frequency of a star on a nearly circular orbit as $\Omega = v_c / R$ and its radial frequency as the epicyclic frequency $\kappa$, then the orbit is closes back on itself if the ratio of $\Omega / \kappa$ is a rational number. More generally, the orbit doesn’t quite close but the point where the radial motion returns to its original position is some constant angle $\delta \theta$ ahead or behind of the starting angle. If we look at the system in a frame which is rotating at a speed such that we move this angle in one rotation of the circular orbit (i.e. at a frequency given by $\Omega_p = \Omega - n\kappa / m$ where $n$ and $m$ are integers), then the orbits do appear to close.

Therefore if we can arrange the orbits to create a pattern (e.g. a spiral pattern), at one epoch then this pattern will rotate with a frequency given by $\Omega_p$ above. Of course, this only works if $\Omega_p$ is independent of $R$ so that the pattern at all radii moves with the same angular frequency. Remarkably, for typical spiral galaxy rotation curves this is true over a wide range of radii for selected values of $n$ and $m$. In particular, for $n = 1$ and $m = 2$ this holds; these orbits generate elliptical orbits and a slow rotation of the position angle of the ellipse with radius creates a two-armed spiral (see Figure 6-11 of Binney & Tremaine for an example of this).

This discussion is purely kinematical, it does not take into account the fact that the axisymmetric perturbations will change the potential and so modify the result. However, a more detailed investigation shows that a process known as swing amplification can actually help grow a small spiral pattern into a more pronounced one. While this effect is clearly important in many real galaxies, it is also quite possible that spiral structure is related to other physical processes, including the increased star formation created by the density enhancements.

### 3.5 Scaling relations: The Tully-Fisher relation and the Virial Theorem

Much like the Fundamental plane for elliptical galaxies, the Tully-Fisher relation is a correlation between the galaxy’s luminosity and the maximum rotational velocity of the gas, often measured via the HI 21 cm line width:

$$L \propto V_{\text{max}}^\alpha$$

where $\alpha \sim 4$ (although the value appears to depend on the wavelength at which the luminosity is measured).

This is strongly reminiscent of the Faber-Jackson relation, implying some common origin. Because we are dealing with self-gravitating, equilibrium systems, with turn to the virial theorem which provides a relation between the kinetic energy (i.e. velocity) and potential energy (which is related to mass or luminosity) for such systems. The virial theorem, which can be derived from the Boltzmann equation (by first multiply by the velocity and integrating over velocities to get the Jeans equation and then multiplying by position and integrating over all positions), can be stated as: $W = -2K$, where we define the potential and kinetic energies as:

$$W = -\frac{GM}{R_g} \quad K = \frac{1}{2} M \langle v^2 \rangle$$

To relate the mass $M$ to the luminosity $L$ of the galaxy, we use the usual definition $\Upsilon = M/L$ and in addition define $C_v = \langle v^2 \rangle / v_{\text{max}}^2$ so that

$$L = \frac{R_g C_v}{G \Upsilon} v_{\text{max}}^2$$
If we assume that the radius $R_g$ is linearly related to the scale length of the disk and that all spiral galaxies have the same surface brightness $I_0$ so that $L = \pi R_g^2 I_0$, then we get:

$$L = \frac{C_v^2}{\pi I_0 G^2 \Upsilon^2} V_{\text{max}}^4$$

which reproduces the scaling of the Tully-Fisher relation. While this appears successful, it hides a number of serious problems. The first is that the surface brightness of all galaxies is not constant – in fact, the scatter in the surface brightness of spirals is several orders of magnitude. However, the Tully-Fisher relation itself is very tight and so we have created something that is more than the sum of its parts: the final relation has a better correlation than its individual components.

Another problem is that rotation velocity curves are set in large part by the dark matter component (because the $v_c$ curves are flat, the dark matter must dominate beyond a few disk scale lengths). This means the amount of dark matter inside a few disk scale lengths must correlate with the mass of baryons. This is, at face value, a surprising result because the baryon distribution is set by its specific angular momentum content (i.e. when rotation balances gravity), while the dark matter structure is presumably set by something else (e.g. the velocity dispersion of the dark matter).

Despite these difficulties, it seems likely that we are on to important element explaining this relation. We can apply the same virial reasoning that we just used on the Tully-Fisher relation to the elliptical galaxies’ Fundamental plane. For example, we can again take the scalar virial theorem and define dimensionless quantities relating physical quantities to observables: $C_v = \langle v^2 \rangle / \sigma_0^2$ and $C_R = R_g / R_e$ where $\sigma_0$ is the central velocity dispersion of the elliptical galaxies. This gives us

$$R_e = C_v C_{\Upsilon} \sigma_0^2 G \pi \langle I \rangle_e,$$

where we have related the gravitating mass to the luminosity via the mass-to-light ratio $\Upsilon$ and employed the mean surface brightness inside the effective radius defined in the section on ellipticals. This bears some similarity to the Fundamental plane as we can see by taking the log of the above expression and writing the surface brightness in terms of magnitudes ($\mu$):

$$\log R_e = \log \frac{C_v C_R}{G \pi \Upsilon} + 2 \log \sigma_0 + 0.4 \mu - 0.4 \log \Upsilon.$$  

The 0.4 exponent in front of $\mu$ is close to the observed 0.36, but $\sigma_0^2$ differs quite a bit from $\sigma_0^{1.4}$. Possible explanations are non-homology (i.e. the structure of the halo does not scale so $C_v$ is not really a constant but depends on $\sigma_0$), or “tilt”, an expression indicating the $\Upsilon$ varies with elliptical mass (or $\sigma_0$).

### 3.6 The Black-hole galaxy relation

Recently, it has become clear that all local galaxies with significant spheroidal components host supermassive blackholes in their centers. Even more remarkably, there is a fairly tight relation between the mass of the black-hole and the mass of the bulge. This (“Magorrian”) relation indicates that roughly 0.5% of the mass of the bulge is in the supermassive blackhole. It is sometimes written differently, in terms of the black-hole mass $M_{\text{BH}}$ and the velocity dispersion of the bulge or spheroid:

$$M_{\text{BH}} \propto \sigma^{4.5}.$$  

It is clear that these blackholes where quasars at some point in the past. In fact, assuming that these black holes grow from the inspiral of gas in an accretion disk accompanied by the emission of roughly 10% of the gravitational energy of the infalling mass (as General Relativity predicts), then the resulting radiative energy agrees well with the total energy that quasars are observed to produce. This comparison is known as the Soltan argument.

The fact that the mass of black holes is tightly correlated to the bulge mass tells us that black-holes are closely connected to their large halos. While it is not clear which way causality runs in this case (i.e. do galaxies control how much gas is fed onto black holes or do quasar outflows regulate star formation in spheroids), it is not hard to imagine that the vast amounts of energy involved in quasars must have a substantial impact on the galaxies that host them.
4 Distribution of galaxies in space

The large-scale distribution of galaxies in space is a very useful probe of cosmology because it is a reflection of the source of inhomogeneity in the early universe. Although we will largely stay away from large-scale structure in these notes, distribution statistics are important to us for two reasons. First, in order to convert the distribution of the galaxy to that of the underlying matter, we need to understand what is the relation between mass and galaxies (sometimes called the bias). Secondly, galaxies are themselves built up from inhomogeneities from the early universe so we need to know at least a little about this topic in order to understand how galaxies come together.

The first topic in this section should properly be a discussion of distance measures, in particular a discussion of the so-called distance ladder and how we actually work out the distance to extragalactic objects. However, this would take us too far outside the main development of the course and so we just refer readers to other source (for example, chapter 7 of Binney & Merrifield’s Galactic Astronomy).

4.1 The Correlation Function

The correlation function quantifies the clustering properties of galaxies above and beyond a uniform distribution. More precisely, the two-point correlation function \( \xi(r) \) is defined via the probability of finding two galaxies separated by a distance \( r \) in volume elements \( dV_1 \) and \( dV_2 \):

\[
dP(r) = n_0^2 [1 + \xi(r)] dV_1 dV_2,
\]

where \( n_0 \) is the mean density of galaxies and we have assumed the distribution is statistical isotropic. We can relate this to the overdensity distribution \( \delta(x) \) through the definition \( n(x) = n_0 [1 + \delta(x)] \). Given this density distribution of galaxies, we can find the probability of finding a galaxy at \( x \) AND at \( x + r \):

\[
dP(x, r) = n_0^2 [1 + \delta(x)][1 + \delta(x + r)] dV_1 dV_2.
\]

Averaging this over all positions \( x \) and assuming statistical isotropy, we get:

\[
dP(r) = \langle dP(x, r) \rangle = n_0^2 [1 + \langle \delta(x)\delta(x + r) \rangle] dV_1 dV_2,
\]

since averages over the overdensity \( \langle \delta(x) \rangle \) are zero by definition. This leaves us with a definition for the two-point correlation function in terms of averages over all positions:

\[
\xi(r) = \langle \delta(x)\delta(x + r) \rangle
\]

The observed galaxy two-point correlation function is well-fit by a power-law:

\[
\xi_{\text{gal}}(r) = (r/r_0)^{-1.8},
\]

where \( r_0 \) is the so-called correlation length and for typical bright galaxies is approximately \( 5h^{-1}\text{Mpc} \). Note that the correlation function depends on the objects being correlated (of course), with elliptical galaxies and clusters of galaxies being more highly correlated.

Note that even for an isotropic distribution, the two-point distribution does not contain all of the possible statistical information. Higher-order correlation function (three-point, etc) can be similarly defined and measured. For a Gaussian random field (more on this later), all of the higher terms can be expressed in terms of the two-point function, so it contains all of the statistical information of such a field.

4.2 The power spectrum

It is often useful to discuss correlation in terms of the Fourier transform of the real overdensity field \( \delta(x) \):

\[
\delta(x) = \frac{V}{(2\pi)^3} \int \delta(k) e^{-ik \cdot x} d^3k.
\]
This is the usual Fourier transform definition, where \( V = L^3 \) is the volume of the box and the inverse transform is given by:

\[
\delta(k) = \frac{1}{V} \int \delta(x)e^{i\mathbf{k} \cdot \mathbf{x}}d^3\mathbf{k}. \tag{35}
\]

We have used integrals in these expressions and so assumed that the box is arbitrarily large. Similar results apply for a finite sized periodic box with integrals replaced by sums. Keep in mind that the Fourier modes are plane waves in real space so each \( \delta(k) \) is telling us the amplitude of an infinite plane wave. The superposition of many of these waves results in the observed real-space distribution of the overdensity field.

To relate this to the correlation function is straightforward. We evaluate equation (32) with the definition given above. To start, we write:

\[
\delta(x)\delta(x + r) = \frac{V^2}{(2\pi)^6} \int \delta(k)e^{-i\mathbf{k} \cdot \mathbf{x}}d^3\mathbf{k} \int \delta(k')e^{-i\mathbf{k}' \cdot \mathbf{x}}e^{-i\mathbf{k}' \cdot \mathbf{r}}d^3\mathbf{k}'
\]

\[
= \frac{V^2}{(2\pi)^6} \int d^3\mathbf{k}d^3\mathbf{k}'\delta(k)\delta(k')^* e^{-i(k-k') \cdot \mathbf{r}}e^{i\mathbf{k}' \cdot \mathbf{r}} \tag{36}
\]

In the second line, we have re-arranged the integral. We have also replaced the integral of \( k' \) by its negative and exploited the fact that the \( \delta(x) \) function is real so that we can use the symmetry relation \( \delta(-\mathbf{k}) = \delta(\mathbf{k})^* \), where the asterisk indicates complex conjugation. The next step is to perform the average operation. Recall that the average is given by \( \langle \ldots \rangle = \frac{1}{V} \int \ldots d^3x \) and we can interchange the integrals over \( x \) and \( k \) to get:

\[
\xi(r) = \langle \delta(x)\delta(x + r) \rangle = \frac{V}{(2\pi)^6} \int \int \delta(k)\delta(k')^* e^{i\mathbf{k}' \cdot \mathbf{r}} \int e^{-i(k-k') \cdot \mathbf{x}}d^3x d^3\mathbf{k}d^3\mathbf{k}' \tag{37}
\]

The integral over space is zero unless \( k = k' \) and otherwise it is equal to \( (2\pi)^3 \) (i.e. it is a delta function), so the expression simplifies considerably to:

\[
\xi(r) = \frac{V}{(2\pi)^3} \int \delta^2(k)e^{i\mathbf{k} \cdot \mathbf{r}}d^3\mathbf{k} \tag{38}
\]

The quantity \( \delta^2(k) \) is the power spectrum and we now see that the correlation function and the power spectrum are simply Fourier Transforms of each other. As usual, we can assume statistical isotropy and so only the magnitudes of the vectors \( \mathbf{k} \) and \( \mathbf{r} \) are important. Since \( P(k) = \delta^2(k) \) only depends on the magnitude \( k \), we can do some of the integral in the previous expression to get:

\[
\xi(r) = \frac{V}{2\pi^2} \int P(k)\frac{\sin kr}{kr}k^2dk \tag{39}
\]

where we have used the fact that we are only interested in the real part of the transform.