Problem 1: The 1D wave equation for an adiabatic gas

The fluid equations (neglecting gravity and assuming a simplified equation of state) are given by:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} &= - \frac{1}{\rho} \frac{\partial P}{\partial x} \\
P &= K \rho^\gamma
\end{align*}
\]

Show that we can derive a wave equation from these equations if we assume the gas can be described by small perturbations away from a uniform, static state. In particular, we assume a solution of the form:

\[
\begin{align*}
\rho(x, t) &= \rho_0 + \rho_1(x, t) \\
u(x, t) &= u_1(x, t) \\
P(x, t) &= P_0 + P_1(x, t),
\end{align*}
\]

where the perturbations are assumed to be small (\(\rho_1 \ll \rho_0, P_1 \ll P_0\)) so that we can neglect second-order terms. Note also that \(\rho_0\) and \(P_0\) are constant in both \(x\) and \(t\). Show that these equations can be combined to produce a wave equation having the form

\[
\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \frac{\partial^2 \rho_1}{\partial x^2}
\]

where \(c_s^2 = \frac{\partial P_1}{\partial \rho_1}\) is the sound speed. Find an explicit expression for the sound speed of an ideal isothermal gas (i.e. \(K = kT/\mu\) where \(T\) is a constant, and \(\gamma = 1\)).

Problem 2: The Constant density stellar model

It is relatively straightforward to solve the equations of stellar structure for a star with a constant density \(\rho_0\). Assuming that you know the total radius of the star \(R\), and its mass \(M\), find the pressure as a function of radius \(r\) and mass shell \(M_r\) (using the equation of hydrostatic equilibrium). You will need to assume that the external pressure vanishes (i.e. \(P = 0\) at \(r = R\)). Derive an expression for the central temperature assuming an ideal gas with \(P = nkT = \rho kT/\mu\). Verify that the Virial theorem is satisfied for this star.

Problem 3: A Lower limit to the central pressure

We can derive a lower limit to the central pressure using the following steps. Consider the function

\[
f(r) = P(r) + \frac{GM_r^2}{8\pi r^4}.
\]
First, show that \( f(r) \) decreases outward with increasing \( r \) (you could do this by demonstrating that \( df/dr < 0 \)).

Next, assuming zero pressure at \( R \), demonstrate (almost immediately) that

\[
P_c > \frac{GM^2}{8\pi R^4}.
\]

Note that you must show \( M_r^2/r^4 \to 0 \) as \( r \to 0 \).

**Problem 4: A slightly improved stellar model**

A bit (but not much) better approximation may be obtained by assuming that the density is a linear fraction of radius:

\[
\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)
\]

where \( \rho_c \) is the central density and \( R \) is the total radius. Assume the usual outer boundary conditions such that \( P(R) = 0 \)

1. Find an expression for the central density in terms of \( R \) and \( M \).

2. Use the equation of hydrostatic equilibrium to find the pressure as a function of radius. Your answer will be in the form of \( P(r) = P_c \) times a polynomial in \( (r/R) \). What is \( P_c \) in terms of \( M \) and \( R \)? (It should be proportional to \( GM^2/R^4 \) – why?). Express \( P_c \) numerically with \( M \) and \( R \) in solar units.

3. Find the central temperature \( T_c \) (assuming an ideal gas). Compare this result to the constant density model (problem 2) – why is the central pressure higher for the linear model while the central temperature is lower?