Problem 1: Galaxy Clusters

Some galaxies are found in “Clusters” of galaxies, with large numbers of galaxies occupying a small region of space. For example, the Coma cluster contains approximately 1000 galaxies in a spherical volume of about 3 Mpc. Imagine that these galaxies are equally distributed within this region (in fact they are more concentrated towards the center, but we’ll ignore that for this problem). The total mass of the cluster is about $10^{15} M_\odot$. You may assume the cluster is in equilibrium.

(a) Calculate the relaxation time $t_{\text{relax}}$ for this cluster of galaxies, treating each $10^{12} M_\odot$ galaxy as an individual “particle”. Is this clustered “relaxed” (and if yes, what do you expect to see)?

(b) Calculate the collision time (i.e. time between physical collisions) for galaxies in this cluster. You can treat the galaxies as if they were spheres with radii of 10 kpc. Do you expect collisions to be important (and if yes, what do you expect to see)?

Problem 2: Dark matter profiles

(a) Show that rigid-body rotation (i.e. constant angular velocity) near the galactic center is consistent with a spherically symmetric mass distribution with constant density.

(b) It has been suggested that the following formula describes the distribution of dark matter in the galaxy:

$$ \rho(r) = \frac{\rho_0}{1 + (r/a)^2} $$

Here, $\rho_0$ and $a$ are constant parameters ($\rho_0$ is the density in the center and $a$ is the core radius of the distribution). Is this consistent with rigid-body rotation near the galactic center ($r < a$), and why or why not?

(c) Recently, based on computer simulations, a new dark matter density profile has been proposed, known as the NFW profile after the initials of the three researchers who first suggested it:

$$ \rho_{\text{NFW}}(r) = \frac{\rho_0}{(r/a)(1 + r/a)^2} $$

Show that the circular velocity of stars in this dark matter halo is given by

$$ V_c^2(r) = C \left[ \ln \left( \frac{1 + r/a}{r/a} \right) - \frac{1}{1 + r/a} \right], $$

and find the constant $C$ in terms of $G$, $\rho_0$ and $a$.  

(over)
Problem 3: The velocity-luminosity relation of spiral galaxies

Recall that the Tully-Fisher relation for spiral galaxies tells us that there is a correlation between a galaxy’s luminosity \( L \) and its maximal circular velocity \( v_{\text{max}} \) (i.e. the maximum of \( v_c(R) \), the rotation velocity curve):

\[
L \propto v_{\text{max}}^4
\]  

(1)

(a) Let’s assume that the spiral galaxies have no dark matter so that the circular velocity comes entirely from the mass in stars. We’ll also assume that all the stars are in a thin exponential disk with surface density \( \Sigma(R) = \Sigma_0 \exp\left(-R/R_d\right) \) as in a previous problem set. Find an expression for the circular velocity just due to stars. You do not have to evaluate the integral, but do write the integral in dimensionless form (i.e. define \( x = R/R_d \)). It turns out that this expression has a maximum around \( R \approx 1.8R_d \) (you can confirm this graphically if you wish). Based on this, it is easy to see that the mass of the galaxy scales as:

\[
M \propto v_{\text{max}}^2R_d.
\]

(b) If the surface brightness of the disk of stars is given by \( I(R) = I_0 \exp\left(-R/R_D\right) \), show that the total luminosity of the galaxy is related to the central surface brightness \( (I_0) \) by \( L = 2\pi I_0 R_d^2 \), where \( I_0 \) is essentially the luminosity per unit area (like \( \Sigma_0 \) was mass per unit area).

(c) Now let’s introduce dark matter. Let’s assume that the total mass and luminosity are proportional (this is the simplest assumption) so that \( M = \Gamma L \), where \( \Gamma \) is called the mass-to-light ratio. Based on the previous results, find a relation between \( L \) and \( v_{\text{max}} \) and show that \( \Gamma \propto I_0^{-1/2} \) if the Tully-Fisher relation is still to hold.

This last result tells us there is something very surprising going on, because we know that \( I_0 \) varies from galaxy to galaxy (\( I_0 \) just tells us how bright the center of the galaxy is), so that \( \Gamma \), the mass-to-light ratio, must adjust itself to fit the relation above. However, we know that in reality most of the mass in a galaxy is dark matter, so the amount of dark matter must depend on how bright the center of the galaxy is. There is no obvious reason why this should be so, and remains one of the biggest mysteries about dark matter in galaxies.