1. Using Shapley’s assumption that M101 has a diameter of 100 kpc, and adopting the (incorrect) observations of Adrian van Maanen (in ∼1910) that indicated M101 was rotating at 0.02" yr$^{-1}$, estimate the speed of a point at the edge of M101. Compare this speed to the characteristic rotation speed of the Milky Way. [Carroll & Ostlie problem 23.1]

2. A star is performing radial orbits through the center of a star cluster which can be approximated by a uniform sphere with radius $R$ and density $\rho$. Using Newton’s laws, show that the time it would take for a star to get back to its original location is $(3\pi/G\rho)^{1/2}$. This is the dynamical time of a system, because it is the characteristic time that stars take to complete an orbit (approximately true even for non-radial orbits or if the density of the system is not uniform).

3. A large number of stars are placed randomly within a spherical region with radius $R$ so that the initial density of stars within the sphere is uniform, and the total mass of stars is $M$. These stars have zero initial velocities, and will immediately begin to collapse due to their own self-gravity. Some time later, the stars are in equilibrium. You may assume that once again the distribution of stars can be described by a uniform sphere and use without proof the fact that the total potential energy for a uniform sphere is

$$PE = \frac{-3GM^2}{5R}$$

Show that after the collapse, when the system is once again in equilibrium, the mean density of stars will have increased by a factor of 8 compared to the initial density of stars.

4. Recall that the Tully-Fisher relation for spiral galaxies tells us that there is a correlation between a galaxy’s luminosity $L$ and its maximal circular velocity $v_{\text{max}}$ (i.e. the maximum of $v_c(R)$, the rotation velocity curve):

$$L \propto v_{\text{max}}^4$$

In this problem we want to see if we can explain this relation.

(a) Let’s assume that the spiral galaxies have no dark matter so that the circular velocity comes entirely from the mass in stars. We’ll also assume that all the stars are in a thin exponential disk with surface density $\Sigma(R) = \Sigma_0 \exp(-R/h_R)$ (the surface density is the mass per unit area rather than the traditional density which is mass per unit volume). $\Sigma_0$ and $h_R$ are the surface density in the center, and the scale length of the disk, respectively,
and both are given parameters. Find an expression for the circular velocity \( v_c(R) \) just due to stars in this disk. Accurately plot this function, showing that it has a maximum around \( R \approx 1.8 h_R \). Based on this, argue that the mass of the galaxy scales as:

\[
M \propto v_{\text{max}}^2 h_R
\]

(b) If the surface brightness of the disk of stars is given by \( I(R) = I_0 \exp(-R/h_R) \), show that the total luminosity of the galaxy is related to the central surface brightness \( (I_0) \) by \( L = 2\pi I_0 h_R^2 \), where \( I_0 \) is essentially the luminosity per unit area (like \( \Sigma_0 \) was mass per unit area).

(c) Now let’s introduce dark matter. Let’s assume that the total mass and luminosity are proportional (this is the simplest assumption) so that \( M = \Gamma L \), where \( \Gamma \) is called the mass-to-light ratio. Based on the previous results, find a relation between \( L \) and \( v_{\text{max}} \) and show that \( \Gamma \propto I_0^{-1/2} \) if the Tully-Fisher relation is still to hold.

This last result tells us there is something very surprising going on, because we know that \( I_0 \) varies from galaxy to galaxy (\( I_0 \) just tells us how bright the center of the galaxy is), so that \( \Gamma \), the mass-to-light ratio, must adjust itself to fit the relation above. However, we know that in reality most of the mass in a galaxy is dark matter, so the amount of dark matter (i.e. \( \Gamma \)) must depend on how bright the center of the galaxy is. There is no obvious reason why this should be so, and remains one of the biggest mysteries about dark matter in galaxies.