**Astronomy 3602: Homework #4**

**Due Wednesday, November 17**

**Problem 1: A Dark Matter "Model" (30 points).**

Assume that the dark matter is composed of a population of iron asteroid balls that fill space uniformly. Each iron ball is identical and has an internal density $\rho_0 = 8$ g cm$^{-3}$, and a radius $r$. For simplicity, in this problem, assume a matter dominated universe with $\Omega_{m,0} = 1$ and $H_0$ =70 km/s/Mpc.

(a) What fraction of the sky would be covered by the collection of asteroids between us ($z = 0$) and a source at redshift $z_s$ ? Your answer will depend explicitly on $r$. (Hint: this is similar to problem 2(a) in Problem Set #1, except that here both the space density of objects, and the volume per unit redshift, evolves - you must include these cosmological corrections.)

(b) What is the lower limit on $r$ from the condition that the universe is not opaque; i.e. that we can see a quasar at redshift $z_s = 5$?

**Problem 2: Dark Matter - 1933 vs. now (20 points).**

Zwicky (1933) compared the mass–to–light (M/L) ratio of the Coma cluster as measured from the virial theorem with the M/L ratio of the luminous parts of individual galaxies as measured by rotation curves. He concluded that there was 400 times as much dark matter as luminous matter in the Coma cluster. However, Zwicky's conclusion was based on the erronous value of the Hubble constant, $H_0$ =558 km/s/Mpc. How is Zwicky's conclusion about the ratio of dark to luminous matter affected, now that we know that the correct value of the Hubble constant is much smaller, $H_0$ =70 km/s/Mpc?

**Problem 3: Potential Energy of a Galaxy Cluster [25 points]**

The total gravitational potential energy of a lump of mass $M$ in general is $E_{\rm pot} = -\alpha GM^2/R$, where $R$ is a characteristic size of the lump and $\alpha$ is a dimensionless constant of order unity. The density distribution in galaxy clusters can be approximated as spherically symmetric, with a radial profile $\rho(r) = A/r^2$ extending out to the edge of the cluster at radius $R$ (here $A$ is a constant). Compute $\alpha$ for this distribution. What is the total energy of the cluster if it is in virial equilibrium? What would be the velocity of a galaxy (which you may regard as a massless test particle) placed in a circular orbit at a distance $r$ from the center of the cluster? (This distribution is often referred to as the "singular isothermal sphere").

**Problem 4: Redshifting of the CMB (25 points)**.

For black-body radiation, the number density of photons in the frequency range between $f$ and $f + df$ is given by the Planck function, which is a function only of the temperature $T$,

$$n(f,T)df = \frac{8\pi}{c^3}\frac{f^2 df}{\exp(hf/kT) - 1}, \tag{1}$$

where $h$ is Planck's constant, $c$ is the speed of light, and $k$ the Boltzmann constant (e.g. see equation 2.30 in Ryden). Note that the dimension of $n(f)$ is $(\text{volume})^{-1}\times(\text{frequency})^{-1}$, or its units are, e.g., $\text{cm}^{-3}\times\text{Hz}^{-1}$.

Normally, this distribution is a result of thermodynamical equilibrium, which must be established by rapid processes that can absorb and emit photons. In this problem, you will show that the expansion of the universe preserves this form exactly, even if the photons are just passive and do not interact with anything. This important property of the FRW universe means that the Planckian shape of the present-day CMB, which has a very low temperature, does *not* imply present equilibrium. Rather, the CMB is 'fossil evidence' of equilibrium in the distant past.

To show this, assume that Eq. (1) holds at some initial early time $t_i$ in the universe, at redshift $z_i$, when the temperature was $T_i$, i.e. $n(f,t_i) = n(f,T_i)$. Consider photons located in some initial volume $V_i$ and in an initial frequency range between $f_i$ and $f_i + df_i$. Now, let's follow these photons to some later time $t_f > t_i$, when the redshift is $z_f < z_i$.

(a) What is the range of frequencies in which these photons are found at $t_f$ ? Express the result in terms of $z_f, z_i, f_i$, and $df_i$.

(b) What is the volume in which these photons are found at $t_f$ ? Express the result in terms of $z_f, z_i$, and $V_i$.

(c) Based on (a) and (b), show that the number density of photons per unit volume and unit frequency range is again given by Eq. (1), but with a re-defined temperature of $T_f = [(1+z_f)/(1+z_i)]T_i$.

This result agrees with the temperature-evolution of a radiation-dominated universe we obtained earlier from the fluid equation, but now you have shown that the *shape* of the photon spectrum remains a 'fake' blackbody!

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