Astronomy GR6001: Problem Set #2

Due in class on Wednesday, October 6, 2021

Problem 1 (20 points):

The detailed balance argument in the text shows that with an appropriate temperature-independent relation between the three Einstein coefficients, the radiation spectrum assumes the Planck shape B_ν. Show that if you apply the same detailed balance argument, but neglect stimulated emission (as classical physicists did before the turn of the last century), the spectrum can be consistent only with the Wien limit of the Planck function B_ν. What would the relation between the emission and absorption coefficients have to be to predict correctly this limiting form of the blackbody spectrum? Would the energy density be finite, if this limit of the Planck function was applied at all frequencies?

Problem 2 (25 points):

Consider a star that is optically thick, so that it is emitting radiation only from a layer near its surface. Assume that this layer is thin compared to the radius of the star - i.e., a plane parallel atmosphere. In this problem, you will calculate the specific intensity emerging from the star along a given ray. Let us measure distance along a ray, using the monochromatic optical depth τ_ν, defined to increase *into* the star (note that this differs from the sign convention used in the equations in the text). Suppose that the source function can be described by $a_\nu + b_\nu \tau_\nu$, where a_ν and b_ν are (positive) constants, neglecting terms of order $O(\tau^2)$. This is known as the *Eddington-Barbier approximation.*

(a) Solve the radiative transfer equation to show that the emergent intensity, in the direction perpendicular to the star's surface, is $I_\nu = S_\nu(1)$.

(b) Repeat the above calculation, for a ray that makes an arbitrary angle θ with respect to the normal to the stellar surface. Express your result as a function of a_ν, b_ν, and $\mu = \cos\theta$. Assuming that the temperature gradient is positive (temperature increasing inward), show that your result produces "limb darkening".

(c) Integrate I_ν appropriately over all directions, to find the total emergent specific flux F_ν, as measured by an observed outside the atmosphere, in terms of a_ν and b_ν. Compare your result to the flux emerging from an isotropic emitter: what is the effective optical depth from which the flux is escaping ? (This is a good approximation of the continuum photospheric depth of the Sun).

Problem 3 (25 points):

Consider a ray passing through two discrete patches of matter (otherwise propagating in vacuum). Both patches contain thermally emitting, homogeneous material, but with different temperatures (T_1 and T_2) and optical depths (τ_1 and τ_2) in the two patches. The ray first enters patch 1, with the incident intensity $I_\nu(0) = 0$. What is the specific intensity $I_{\rm obs}$, observed to emerge from patch 2? Under what conditions do you expect the observed brightness temperature to be close to T_1? To T_2?

Problem 4 (30 points):

Before the epoch of reionization, neutral hydrogen in the universe was interacting with the cosmic microwave background (CMB). In this problem, you will compute the spin temperature of the 21cm line, assuming that the level populations are determined both by atomic collisions and interactions with the CMB. Recall that the spin temperature $T_{\rm S}$ is defined as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} \exp\left(-\frac{h\nu}{kT_{\rm S}}\right). \tag{1}$$

Assuming equilibrium, detailed balance between the level populations n_1 and n_0 obey

$$n_0(B_{01}B_\nu + nk_{01}) = n_1(A_{10} + B_{10}B_\nu + nk_{10}), \tag{2}$$

where $n = n_0 + n_1$ is the total hydrogen number density, B_ν is the specific intensity of the CMB (which at redshift z has a black-body shape with $T_{\rm CMB} = 2.73(1+z)^\circ$K), and k_{01} and k_{10} are the collisional excitation and deexcitation rate coefficients. Detailed balance in kinetic equilibrium in the absence of radiation imposes the relation $n_0 k_{01} = n_1 k_{10}$, so that

$$\frac{k_{01}}{k_{10}} = \frac{g_1}{g_0} \exp\left(-\frac{h\nu}{kT_{\rm g}}\right), \tag{3}$$

where T_g is the gas temperature. Show that the spin temperature is

$$T_{\rm S} = \frac{T_{\rm CMB} + yT_{\rm g}}{1+y}, \tag{4}$$

where y is a collisional coupling parameter that depends on n and $T_{\rm g}$, $y = \frac{h\nu}{kT_{\rm g}}\frac{n}{n_{\rm cr}}$. Here k is the Boltzmann consant and $n_{\rm cr} = A_{10}/k_{10} = 3 \times 10^{-5}$ cm^{-3} is the critical density between radiative and collisional deexcitation. You may use the fact that $h\nu \ll kT_{\{\rm S,g,CMB\}}$. Note that at low density $T_{\rm S} \to T_{\rm CMB}$, whereas at high density, collisions couple the spin to the gas temperature, $T_{\rm S} \to T_{\rm g}$. Evaluate the spin temperature at $z = 20$ for the background universe, assuming that the neutral hydrogen density is $n = 10^{-7}(1+z)^3$ cm^{-3}. Before stars formed, the gas temperature followed the CMB temperature (due to Compton scattering) down to redshift $z = 200$ [i.e. $T_{\rm g}(z = 200) = T_{\rm CMB}(z = 200)$], and subsequently cooled adiabatically ($T_{\rm g} \propto (1+z)^2$ at $z \leq 200$). Is the sky dimmer or brighter at the wavelength $\lambda_{\rm obs} = 21(1+z)$cm than the CMB at this wavelength in the absence any neutral hydrogen?